Fast Finite-Time Consensus Tracking of Second-Order Multi-Agent Systems with a Virtual Leader

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Abstract—This paper proposes a new finite-time consensus tracking protocol for reaching the fast finite-time consensus tracking. The Lyapunov function method, algebra graph theory, and homogeneity with dilation are employed to obtain the convergence criteria. Numerical simulations show that compared with the traditional finite-time consensus tracking protocols, the proposed protocol can accelerate the convergence speed of achieving the finite-time consensus tracking.

Index Terms—Second-Order Multi-Agent Systems; A Virtual Leader; Fast finite-Time Consensus Tracking; Convergence Speed

I. INTRODUCTION

During the past decade, distributed coordinated control of multi-agent systems due to its broad applications has received considerable attention in various fields such as physics, biology, computer science and control engineering. In the distributed coordinated control of multi-agent systems, a critical problem is to find proper control protocols to enable all agents to reach an agreement on certain quantities of interest, which is usually called the consensus problem. For most of the existing consensus protocols, [1-5] the final common value to be achieved is a function of initial states of all agents and is inherently a prior unknown constant. This is the so-called $\chi$-consensus [6].

However, in many practical applications, it is required that all agents communicating with their neighbors eventually converge to a desired reference state. This is the so-called leader-follower consensus or consensus tracking.

In leader-follower multi-agent networks, the leaders are usually independent of their followers, but have influence on the followers’ behaviors. Hence, by controlling only the leaders, the control objective of the networks can be realized easily. This not only simplifies the design and implementation of the controls but also saves the control energy and cost [7]. Up to now a lot of attention has been paid to consensus tracking of leader-follower multi-agent systems [8-22].

However, most of the above references are mainly about finding convergence conditions of achieving the consensus or consensus tracking rather than improving the convergence performance, one of which is the convergence speed. However, the convergence speed is really an important performance index, which affects the real-time performance. In recent years, significant attentions have been paid to how to enhance the convergence speed of multi-agent systems or networks. [23-26] Li and Fang designed the optimal weights associated with edges of undirected graph to make the states of the multi-agent systems converge to consensus with a fast speed as well as the maximum communication time-delay can be tolerated. [23] Zhou and Wang proposed the asymptotic and per-step convergence factors as measures of the convergence speed, and derived the exact value for the per-step convergence factor.[24] Wu and Fang proposed the consensus protocol with delayed-state-derivative feedback, demonstrating that choosing the proper gain of delayed-state-derivative feedback can accelerate the convergence speed. [25] Fang et al. [26] proposed the consensus protocol with weighted average prediction, and proved that choosing the proper length of weighted average prediction can enhance the convergence speed.

Although using the methods in Refs. [23-26] can accelerate the convergence speed of achieving the consensus, the achievement of consensus is asymptotical, which means that the consensus can never be reached in finite time. However, in many situations, it is often required that the consensus be reached in finite time, such as when the control accuracy is crucial. Besides faster convergence, other advantages of finite-time consensus include better disturbance rejection and robustness against uncertainties. [27] Therefore, it is necessary to investigate the finite-time consensus of multi-agent systems. Now, there are a lot of results about finite-time consensus. [28-31] In Ref. [28], Cortes considered the finite-time consensus based on the discontinuous protocol. In Ref. [29] Xiao and Wang gave two continuous consensus protocols to solve the finite-time consensus problems of first-order multi-agent systems. Sun and Guan investigated the finite-time consensus problems of leader-follower second-order multi-agent systems under fixed and switching networks. [30] In Ref. [31], Zhu and Guan investigated the
finite-time consensus problems for heterogeneous multi-agent systems, where the virtual leader can be a first-order or a second-order integrator agent.

Compared with the existing consensus protocols with asymptotical convergence in Refs. [23-26], the protocols in Refs. [28-31] can guarantee the finite-time consensus. However, in some practical applications such as braking systems of multiple autonomous vehicles, the faster finite-time consensus, i.e., the consensus with a shorter setting time, is needed. Therefore, it is significant to study how to accelerate the convergence speed of finite-time consensus. Therefore, the main motivation of this paper is to explore the finite-time distributed tracking control protocol based on the traditional finite-time consensus idea and to acquire a fast convergence speed. To this end, in this paper we propose a novel finite-time consensus tracking protocol to solve the fast finite-time consensus tracking problems of the second-order multi-agent systems.

An outline of the paper is as follows. Some preliminaries are provided and the problem is stated in section II. Convergence analysis of the fast finite-time consensus protocol is given in section III. In section IV numerical simulations illustrate the theoretical results, and in section V conclusions are drawn.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. Algebra Graph Theory

Let \( G = (V, E, A) \) be a weighted undirected graph with a set of nodes \( V = \{v_1, v_2, \ldots, v_n\} \), a set of edges \( E \subseteq V \times V \), and the weighted adjacency matrix \( A = [a_{ij}] \) with nonnegative adjacency elements \( a_{ij} \). The node indexes of \( G \) belong to a finite index set \( I = \{1, 2, \ldots, n\} \). An edge of \( G \) is denoted by \( e_{ij} = (v_i, v_j) \). The adjacency elements associated with the edges are positive, i.e., \( a_{ij} \in E \Leftrightarrow a_{ij} > 0 \). Moreover, we assume \( a_{ii} = 0 \) for all \( i \in I \). For the undirected graph \( G \), the adjacency matrix \( A \) is symmetric, i.e., \( a_{ij} = a_{ji} \).

The set of neighbors of node \( v_i \) is denoted by \( N_i = \{v_j \in V : e_{ij} \in E\} \). The degree of node \( v_i \) is defined as \( d_i = \sum_{j \in N_i} a_{ij} \). The Laplacian matrix of \( G \) is defined as \( L = D - A \), where \( D = \text{Diag} \{d_1, d_2, \ldots, d_n\} \) is the degree matrix of \( G \) with diagonal elements \( d_i \) and zero off-diagonal elements. An important fact of \( L \) is that all row sums are zero and thus \( L \) has a right eigenvector \( \mathbf{1}_n \) associated with the zero eigenvalue, where \( \mathbf{1}_n \) denotes the \( n \)-dimensional column vector with all elements being equal to 1.

Path and weight. A path between two distinct nodes \( v_i \) and \( v_j \) means a sequence of distinct edges of the form \( (v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_{m-1}}, v_j) \). A graph is called connected if there is a path between any two distinct nodes of the graph. For the leader-follower consensus problem, we consider graph \( G \) associated with the system consisting of \( n \) agents (which are called followers) and one virtual leader denoted by agent 0. Let \( a_{i0} \) be the adjacency weight between agent \( i \) and the virtual leader. Assume that \( a_{i0} = 1 \), if the virtual leader is a neighbor of agent \( i \), and otherwise \( a_{i0} = 0 \).

B. Problem Statement

In a multi-agent system with \( n \) agents, an agent and an available information flow between two agents are considered as a node and an edge in an undirected graph, respectively.

Consider the system of first-order dynamic agents described by

\[
\dot{x}(t) = v(t), \quad i \in I,
\]

where \( x_i(t) \in \mathbb{R} \) is the position state of agent \( i \), and \( u_i(t) \in \mathbb{R} \) is the control input. The dynamics of the virtual leader is

\[
\dot{x}_0(t) = 0
\]

Consider the second-order system of dynamic agents described by

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t), \\
\dot{v}_i(t) &= u_i(t),
\end{align*}
\]

where \( x_i(t) \in \mathbb{R} \) and \( v_i(t) \in \mathbb{R} \) are the position state and the velocity state of agent \( i \), respectively, and \( u_i(t) \in \mathbb{R} \) is the control input. The dynamics of the virtual leader is

\[
\begin{align*}
\dot{x}_0(t) &= v_0(t), \\
\dot{v}_0(t) &= 0,
\end{align*}
\]

where \( i = 1, 2, \ldots, n \), \( N_i \), and \( 0 < \alpha_i < 1 \). \( \| \cdot \| \) denotes the absolute value, \( p_i \) is the adjacency weight between the virtual leader and agent \( i \), and \( \text{sign}(\cdot) \) is the sign function. Under the condition that network graph \( G(A) \) is balanced and the leader is globally reachable, the finite-time consensus tracking of the first-order multi-agent system was investigated.

Moreover, in Ref. [30] Sun et al. proposed the following finite-time consensus tracking protocol for a second-order multi-agent system with \( n \) follower-agents and a virtual leader described by

\[
\begin{align*}
u_i(t) &= \sum_{j \in N_i} a_{ij} \text{sign}(x_j - x_i)^{\alpha_i} - p_i \text{sign}(x_i - x_0)^{\alpha_0} \\
&+ a_{i0} \text{sign}(x_0 - x_i)^{\alpha_0} + a_{i0} \text{sign}(v_0 - v_i)^{\alpha_0}.
\end{align*}
\]
Different from the protocol (6), in this paper we develop a new finite-time consensus tracking protocol for the second-order multi-agent system:

\[
\begin{align*}
    u_i(t) &= \sum_{j \in N_i} a_{ij} \text{sig}(x_j - x_i)^\alpha + \sum_{j \in N_i^c} a_{ij} \text{sig}(v_j - v_i)^\alpha \\
    & \quad + a_{ii} \text{sig}(x_i - x_i)^\alpha + a_{ii} \text{sig}(v_i - v_i)^\alpha \\
    & \quad + \gamma \sum_{j \in N_i} a_{ii} (x_j - x_i + v_j - v_i) \\
    & \quad + \gamma a_{ii} (x_i - x_i + v_i - v_i),
\end{align*}
\]  

(7)

where \(0 < \alpha_1 < 1, \alpha_2 = 2 \alpha_1, \alpha_1 + \gamma \geq 0\).

**Definition 1** Leader-follower finite-time consensus is said to be achieved if, there is a setting time \(T_o \in [0, +\infty)\) such that for any initial states, the solution of system (3) satisfies:

\[
\lim_{t \to +\infty} \|x_i(t) - x_o(t)\| = 0, \quad \lim_{t \to +\infty} \|v_i(t) - v_o(t)\| = 0,
\]

and \(x_i(t) = x_o(t), v_i(t) = v_o(t), \quad \forall t \geq T_o, i \in I\).

**III. CONVERGENCE ANALYSIS**

In this section, employing the Lyapunov function method, algebra graph theory, homogeneity with dilation, and some other techniques, we prove that multi-agent system (3) applying the protocol (7) can reach the finite-time consensus tracking.

Before moving on, we need the following assumption and lemmas.

**Assumption 1** The communication network topology \(G\) composed of \(n\) agents is fixed, undirected and connected, and at least one agent has access to the leader.

**Remark 1** Analogous to the analysis in Ref. [30], it is easy to prove that the multi-agent system (3) applying the protocol (6) can achieve consensus in finite time, if the systems (3) and (4) with \((x_1, \ldots, x_n, v_1, \ldots, v_n)\) are homogeneous of degree \(\kappa = \alpha_1 - 1 < 0\) with dilation \((r_1, \ldots, r_n)\) and can achieve consensus asymptotically.

**Lemma 1** [31] Suppose that the function \(\varphi: R^2 \to R\) satisfies \(\varphi(x_i, x_j) = -\varphi(x_j, x_i), i, j \in \Gamma, i \neq j\). Then for any undirected graph \(G\) and a set of numbers \(y_1, y_2, \ldots, y_n\),

\[
\sum_{i=1}^{N} \sum_{j \in N_i} a_{ij} \varphi(x_j - y_i) = -\frac{1}{2} \sum_{i, j \in \varepsilon} a_{ij} (y_i - y_j) \varphi(x_i - x_j).
\]

**Lemma 2** [33] (Lasalle’s Invariance Principle) Let \(x(t)\) be a solution of \(\dot{x} = f(x)\), \(x(0) = x_0 \in R^m\), where \(f: U \to R^m\) is continuous with an open subset \(U\) of \(R^m\), and \(V: U \to R^m\) is a locally Lipschitz function such that \(D^+ V(x(t)) \leq 0\), where \(D^+\) denotes the upper Dini derivative. Then \(\Theta^+(x_0)\) is contained in the union of all solutions that remain in \(S = \{x \in U : D^+ V(x) = 0\}\), where \(\Theta^+(x_0)\) denotes the positive limit set.

Next, the homogeneity with dilation (Rosier 1992) is given for the finite-time convergence analysis. For the \(n\)-dimensional system

\[
\dot{x} = f(x), x = (x_1, x_2, \ldots, x_n) \in R^m,
\]

(8)

a continuous vector field

\[
f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T \text{ is homogeneous of degree } \kappa \in R \text{ with dilation } r = (r_1, r_2, \ldots, r_n), \text{ if for any } \varepsilon > 0,
\]

\[
f_1(\varepsilon \cdot x_1, \varepsilon \cdot x_2, \ldots, \varepsilon \cdot x_n) = \varepsilon^\kappa \cdot f_1(x), i = 1, 2, \ldots, n.
\]

**Definition 2** System (8) is called homogeneous if its vector field is homogeneous. Moreover,

\[
\dot{x} = f(x) + f(x), f(0) = 0, \quad x \in R^m,
\]

(9)

is said to be locally homogeneous of degree \(\kappa \in R\) with respect to the dilation \((r_1, r_2, \ldots, r_n)\), if \(f(x)\) is homogeneous of degree \(\kappa \in R\) with respect to the dilation \((r_1, r_2, \ldots, r_n)\) and \(f\) is a continuous vector field satisfying

\[
\lim_{\varepsilon \to 0} \frac{f(\varepsilon \cdot x_1, \varepsilon \cdot x_2, \ldots, \varepsilon \cdot x_n)}{\varepsilon^\kappa} = 0, \quad \forall x \neq 0, \quad i = 1, 2, \ldots, n.
\]

(10)

For convenience, let \(\text{sign}(x)^\alpha = |x|^{\alpha} \text{sign}(x)\), where \(\text{sign}(\cdot)\) denotes the sign function and \(|a|\) denotes the absolute value of the real number \(a\).

**Lemma 3** [35] Suppose that system (8) is homogeneous of degree \(\kappa \in R\) with dilation \((r_1, r_2, \ldots, r_n)\), the function \(f(x)\) is continuous, and \(x = 0\) is its asymptotically stable equilibrium. Then the equilibrium of system (8) is finite-time stable if the homogeneity degree \(\kappa < 0\). Moreover, the equilibrium of system (9) is locally finite-time stable if (10) holds.

Now, we give the main results.

**Theorem 1** Under Assumption 1, the multi-agent system (3), applying the consensus tracking protocol (7), achieves the finite-time consensus tracking.

**Proof** Define the error vector

\[
\begin{align*}
    \bar{x}_i(t) &= x_i(t) - x_o(t), \\
    \bar{v}_i(t) &= v_i(t) - v_o(t), \quad \forall i \in I.
\end{align*}
\]

(11)

According to Eqs. (3), (4), (7), and (11), we have

\[
\begin{align*}
    \bar{x}_i(t) &= \bar{v}_i(t), \\
    \bar{v}_i(t) &= u_i(t) \\
    &= a_{i0} \text{sig}(\bar{x}_i - \bar{x}_o)^\alpha + \sum_{j \in N_i} a_{ij} \text{sig}(\bar{v}_j - \bar{v}_o)^\alpha \\
    &\quad - a_{i0} \text{sig}(\bar{v}_i - \bar{v}_o)^\alpha - \sum_{j \in N_i^c} a_{ij} \text{sig}(\bar{v}_j - \bar{v}_i)^\alpha \\
    &\quad + \gamma \sum_{j \in N_i} a_{ii} (\bar{x}_j - \bar{x}_i + \bar{v}_j - \bar{v}_i) \\
    &\quad + \gamma a_{ii} (\bar{v}_i - \bar{v}_i).
\end{align*}
\]

(12)
Take the following candidate Lyapunov function
\( V = V_1 + V_2 + V_3 + V_4 + V_5 \) with
\[
V_1 = \frac{1}{2} \sum_{i=1}^{n} \bar{v}_i^2,
\]
\[
V_2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\gamma}{\alpha_{ij}} \sum_{s=1}^{\alpha_{ij}} a_{ij,s} \mu_i(s) \mu_j(s) ds,
\]
\[
V_3 = \sum_{i=1}^{n} \int_{0}^{\infty} a_{ii} \mu_i(s) \mu_i(s) ds,
\]
\[
V_4 = \frac{1}{2} \sum_{i=1}^{n} a_{ii} \bar{v}_i^2,
\]
and
\[
V_5 = \frac{\gamma}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)^2.
\]

Consider the derivative of \( V_i (i = 1, 2, 3, 4, 5) \) along the trajectories of system (12),
\[
\dot{V}_1 = \sum_{i=1}^{n} \bar{v}_i \dot{\bar{v}}_i - \frac{1}{2} \sum_{i=1}^{n} \frac{\gamma}{\alpha_{ii}} \sum_{j=1}^{\alpha_{ii}} a_{ii,j} \mu_i(s) \mu_i(s) ds - a_{ii} \mu_i(s) \mu_i(s) ds - \frac{1}{2} \sum_{i=1}^{n} a_{ii} \bar{v}_i^2.
\]
\[
\dot{V}_2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j) - a_{ij} \mu_i(s) \mu_j(s) ds - \frac{1}{2} \sum_{i=1}^{n} a_{ii} \bar{v}_i^2.
\]
\[
\dot{V}_3 = \sum_{i=1}^{n} a_{ii} \bar{v}_i^2,
\]
\[
\dot{V}_4 = \gamma \sum_{i=1}^{n} a_{ii} \bar{v}_i \dot{\bar{v}}_i,
\]
and
\[
\dot{V}_5 = \frac{\gamma}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j).
\]

Then
\[
\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 + \dot{V}_5
\]
\[
= \sum_{i=1}^{n} \bar{v}_i \dot{\bar{v}}_i - \frac{1}{2} \sum_{i=1}^{n} \frac{\gamma}{\alpha_{ii}} \sum_{j=1}^{\alpha_{ii}} a_{ii,j} \mu_i(s) \mu_i(s) ds - \frac{1}{2} \sum_{i=1}^{n} a_{ii} \bar{v}_i^2
\]
\[
+ \gamma \sum_{i=1}^{n} a_{ii} \mu_i(s) \mu_i(s) ds - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j)
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j) + \gamma \sum_{i=1}^{n} a_{ii} \bar{v}_i^2
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j),
\]
\[
= \sum_{i=1}^{n} \bar{v}_i \dot{\bar{v}}_i - \frac{1}{2} \sum_{i=1}^{n} \frac{\gamma}{\alpha_{ii}} \sum_{j=1}^{\alpha_{ii}} a_{ii,j} \mu_i(s) \mu_i(s) ds - \frac{1}{2} \sum_{i=1}^{n} a_{ii} \bar{v}_i^2
\]
\[
+ \gamma \sum_{i=1}^{n} a_{ii} \mu_i(s) \mu_i(s) ds - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j)
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j) + \gamma \sum_{i=1}^{n} a_{ii} \bar{v}_i^2
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j),
\]

Thus,
\[
\sum_{i=1}^{n} \bar{v}_i \dot{\bar{v}}_i - \frac{1}{2} \sum_{i=1}^{n} \frac{\gamma}{\alpha_{ii}} \sum_{j=1}^{\alpha_{ii}} a_{ii,j} \mu_i(s) \mu_i(s) ds - \frac{1}{2} \sum_{i=1}^{n} a_{ii} \bar{v}_i^2
\]
\[
+ \gamma \sum_{i=1}^{n} a_{ii} \mu_i(s) \mu_i(s) ds - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j)
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j) + \gamma \sum_{i=1}^{n} a_{ii} \bar{v}_i^2
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j)
\]
\[
= \sum_{i=1}^{n} \bar{v}_i \dot{\bar{v}}_i - \frac{1}{2} \sum_{i=1}^{n} \frac{\gamma}{\alpha_{ii}} \sum_{j=1}^{\alpha_{ii}} a_{ii,j} \mu_i(s) \mu_i(s) ds - \frac{1}{2} \sum_{i=1}^{n} a_{ii} \bar{v}_i^2
\]
\[
+ \gamma \sum_{i=1}^{n} a_{ii} \mu_i(s) \mu_i(s) ds - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j)
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j) + \gamma \sum_{i=1}^{n} a_{ii} \bar{v}_i^2
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j)
\]

Note that \( \dot{V} = 0 \) if and only if \( \bar{v}_i = \bar{v}_i = 0 \), then
\( \bar{v}_i = 0, \forall i \in J \), that is
\( \bar{v}_i = \sum_{j \in J} \bar{x}_j - \bar{x}_j = \sum_{j \in J} a_{ij} \mu_i(s) \mu_i(s) ds - \frac{1}{2} \sum_{i=1}^{n} a_{ii} \bar{v}_i^2 \)
\[
+ \gamma \sum_{i=1}^{n} a_{ii} \mu_i(s) \mu_i(s) ds - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j) + \gamma \sum_{i=1}^{n} a_{ii} \bar{v}_i^2 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j) = 0.
\]

At the same time, from Assumption 1, one can get
\[
\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)(\bar{x}_i - \bar{x}_j) + \sum_{i=1}^{n} a_{ii} x_i \mu_i(s) \mu_i(s) ds
\]
\[
+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\bar{x}_i - \bar{x}_j)^2 + \gamma a_{ii} \mu_i(s) \mu_i(s) ds \geq 0.
\]
The inequality (14) together with (13) gives 
\[ x_i - x_j = 0, \quad \forall i \neq j, i, j \in I. \] From Lemma 3, we have 
\[ x_i - x_j \to 0, \quad \forall i \in I, \quad \text{as} \quad t \to \infty. \]

Next, from Remark 1, we know that the systems (3) and (4) under protocol (7) are homogeneous of degree 
\[ \kappa = \alpha - 1 < 0 \] with dilation 
\[ (2, 2 \ldots, 2, \alpha_i + 1, \alpha_i + 1, \ldots, \alpha_i + 1). \]
Therefore, it follows from Lemma 3 that the multi-agent systems (3) and (4) under protocol (7) reach consensus in finite time. The proof is completed.

**Remark 2** It is obvious that if \( \gamma = 0 \), the protocol (7) degenerates into the protocol (6). With the non-zero parameter \( \gamma \), the convergence speed of the multi-agent systems (3) and (4) under the protocol (7) is faster than that under the protocol (6). This improvement of convergence speed will be illustrated by the following comparison simulations.

### IV. SIMULATIONS

In this section, numerical simulations are provided to illustrate the Effectiveness of the above theoretical results. Consider a multi-agent system composed of three agents and one virtual leader labeled as agent 0 with the network topology shown in Fig. 1.

![Network topology composed of three agents and one virtual leader](image1)

**Figure 1.** Network topology composed of three agents and one virtual leader

Without loss of generality, all weights of edges are assumed to be 1 if \( e_i \in E \), and the initial states are chosen as \( x(0) = (2, 5, 8)^T \) and \( v(0) = (10, 4, 1)^T \). Suppose \( \alpha_1 = 0.8, \alpha_2 = 2\alpha_1/\alpha_1 + 1 = 0.8889, \quad x_0 = 2t \) and \( v_0 = 2 \). The numerical results are shown in Figs. 2 and 3, respectively. It can be seen from Fig. 2 that the system (3), applying the consensus tracking protocol (6), achieves the finite-time consensus tracking with \( T_0 = 30s \). From Fig. 3, we find that the system (3), applying the consensus tracking protocol (7) with \( \gamma = 3 \), achieves the finite-time consensus tracking with \( T_0 \approx 10s \). This numerically shows that the proposed finite-time consensus tracking protocol (7) has the faster convergence speed than the finite-time consensus tracking protocol (6).

### V. CONCLUSIONS

In this paper, we have investigated the fast finite-time consensus tracking problems of second-order multi-agent systems. Applying the Lyapunov function method, algebra graph theory, homogeneity with dilation, and some other techniques, we have proved that second-order multi-agent systems applying the proposed consensus tracking protocol can reach the finite-time consensus tracking. Last, comparison simulations verified the effectiveness of the proposed protocol on improving the convergence speed. One of future research directions is to consider the case with time delays.

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