Blow-up in the Parabolic Problems under Nonlinear Boundary Conditions

Jin Li
School of Mathematics and Statistic, Hexi University, Zhangye, Gansu 734000, PR China
Email: lijingfl1956@163.com.

Abstract—The paper deals with a degenerate and singular parabolic equation with nonlinear boundary condition. We first get the behavior of the solution at infinity, and establish the critical global existence exponent and critical Fujita exponent for the fast diffusive equation, furthermore give the blow-up set and upper bound of the blow-up rate for the nonglobal solutions.

Index Terms—Global Existence Curve; Critical Fujita Curve; Nonlinear Boundary Condition; Blow-Up

I. INTRODUCTION

In this paper, we consider the following the degenerate and singular parabolic equation

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \frac{|D^m u|^p}{|D^m u|} \frac{\partial u^m}{\partial x} \right), \quad x > 0, 0 < t < T,
\]

coupled via nonlinear boundary flux

\[
-\left| \frac{\partial u^m}{\partial x} \right| \frac{\partial u^m}{\partial x}(0, t) = u^*(0, t), \quad 0 < t < T,
\]

and initial data

\[
u(x, 0) = u_0(x), \quad x > 0,
\]

where parameters \(0 < m \leq 1\), \(-1 < p < 1 - m\), \(q > 0\), and \(u_0\), is nonnegative continuous function with compact support in \(\mathbb{R}\).

Nonlinear parabolic equation (1) comes from the theory of turbulent diffusion (see [2, 7] and references therein) and appears in population dynamics, chemical reactions, heat transfer, and so on. The equation (1) includes both the medium porosity operator (with \(p = 0\)) and the gradient-diffusionity the p-Laplacian operator (\(m = 1\)) as special cases, which have been the subject of intensive study (see [2, 4, 6, 7, 8, 10, 13, 21, 25, 30, 33] and references therein). In particular, many paper have been devoted to study critical exponents of the slow diffusion case (see [2, 4, 6, 8, 10, 13, 22, 23, 26, 29, 31, 33, 34, 35] and references therein). Recently, many authors transfer their attention to the fast diffusion case (see [4, 8, 9, 14, 15, 17, 28, 32]) and many important results about critical exponents have been obtained. The concept of critical Fujita exponents was proposed by Fujita in the 1960s during discussion of the heat conduction equation with a nonlinear source (see [5]). In [7], Galaktionov and Levine studied the following single equation

\[
u_t = \nabla (|\nabla v|^p \nabla v^m) + v^m, \quad x \in \mathbb{R}^N, \quad t > 0
\]

\[
u(x, 0) = \nu_0(x), \quad x \in \mathbb{R}^N
\]

where \(\sigma > 0\), \(m > 1\), \(p > 1\) and \(\nu_0(x)\) is a bounded positive continuous function. They shown that the critical exponent is \(p_c = m + \sigma + \frac{2 + \sigma}{N}\).

Recently, Jiang [10] studied the following single equation

\[
\begin{cases}
u_t = (|\nu_t|^p (\nu_t^m)), \quad x > 0, 0 < t < T,  \\
- |\nu_t| (\nu_t^m)(0, t) = u^*(0, t), \quad 0 < t < T,  \\
u(x, 0) = \nu_0(x), \quad x > 0,
\end{cases}
\]

where \(m \geq 1\), \(p > 0\), \(q > 0\). They obtained the critical global existence exponent \(q_0 = \frac{2q + m + 1}{q + 2}\) and the critical Fujita exponent \(p_c = 2q + m + 1\). These results are the extensions of those of Galaktionov and Levine [8].

Motivated by the above mentioned works, in this paper, we investigate the critical global existence exponent \(q_0\) and the critical Fujita exponent \(q_c\) of a nonlinear boundary value problem (1)-(3). We obtain the decay behavior of the solution at infinity and establish the same critical exponents. In addition, we show that the critical Fujita exponent \(q_c\) belongs to blow-up case. Furthermore, we give the blow-up set and upper bound of the blow-up rate of nonglobal solutions.

We remark the main difference between \(p \geq 0, m > 1\) and our current settings \(-1 < p < 1 - m\), \(0 < m < 1\), we take \(p = 0\) for example. For the former, equations (1) having \(m > 1\) are the well-known porous medium equations, while for the latter, Eqs. (1) having \(0 < m < 1\) are the so-called fast diffusion equations. The porous medium equations have finite speed of propagation property, that is, solutions with compactly supported initial data stay compactly supported, which makes comparison with global supersolutions easier when one is...
restricted to compactly supported initial data. However, the solutions of the fast diffusion equations shall become instantaneously positive everywhere for any nontrivial nonnegative initial data, and hence we have to take care of the decay of the solutions.

Owing to the degeneration (singularity) of problem (1)-(3), a precise definition of a weak solution is needed.

Definition 1. Let $T > 0$, and denote $Q_T = \Omega \times (0,T]$ with $Q_T = \Omega \times (0,T)$. A positive function $u(x,t) \in C(\overline{Q}_T)$ is called an upper (lower) solution of (1) in $Q_T$ with nonlinear flux $a^u$ if:

1. $u \in L^\infty(0,T;W^{1,\infty}(\Omega)) \cap W^{1,1}(0,T;L^1(\Omega)), u(x,0) \geq (\leq) u_0$; and
2. for any positive function $\omega \in L^1(0,T;W^{1,\infty}(\Omega)) \cap L^1(\overline{Q}_T)$, we have

$$
\int_0^T \int_\Omega \frac{\partial \omega}{\partial t} \left( -\frac{\partial u}{\partial x} + \frac{\partial a^u}{\partial x} \right) dx dt \geq (\leq) \int_0^T \int_\Omega \omega a^u dx dt;
$$

$u(x,t)$ is called a weak solution of (1)-(3) if it is both a weak upper and a lower solution.

The local existence of the weak solution to the problem (1)-(3) and the comparison principle can be easily established (see the survey [12] and books [3, 18, 30]). In order to investigate the blow-up properties of solutions to (1)-(3), we need to study the behavior of $u(x,t)$ for large $x$ and obtain the following result.

In order to investigate the blow-up properties of solutions to (1)-(3), we need to study the behavior of $u(x,t)$ for large $x$ and obtain the following result.

Theorem 1.1. The positive solution of the problem (1)-(3) has, for each $t \in (0,T)$,

$$
\liminf_{x \to \infty} x^{\frac{p+1}{m-p}} u(x,t) \geq \left( C_{m,p}^{-1}(p+1) \right)^{-\frac{1}{m-p}} (7)
$$

where $T$ is the maximal existence time for the solution, which may be finite or infinite, and

$$
C_{m,p} = \frac{1-m-p}{p+2} \left( \frac{1}{m(2p+m+1)} \right) (8)
$$

Now we establish the critical exponents of the problem (1)-(3) as follows.

Theorem 1.2. The critical global existence exponent and critical Fujita exponent for the problem (1)-(3) are given by $q_0 = \frac{2p+m+1}{p+2}$ and $q_c = 2p+m+1$, respectively. More precisely, we have

1. if $0 < q \leq q_0 = \frac{2p+m+1}{p+2}$, every nontrivial and nonnegative solution of the problem (1)-(3) exists globally in time;
2. if $q > q_0 = \frac{2p+m+1}{p+2}$, then the solution of the problem (1)-(3) with appropriately large initial data blows up in a finite time;

(3) if $q_0 = \frac{2p+m+1}{p+2} < q < q_c = 2p+m+1$, every nontrivial and nonnegative solution of the problem (1)-(3) blows up in finite time;

(4) if $q > 2p+m+1$, then the problem (1)-(3) admits nontrivial global solutions with small initial data.

Theorem 1.3. When $q = q_c$, each positive solution blows up in finite time.

Remark 1.1. Theorem 1.3 shows that the critical Fujita exponent $q_c$ belongs to the blow-up case.

Theorem 1.4. Let $B(u)$ be the blow-up set of the solution $u$ to (1)-(3), then $B(u) = \emptyset$.

To get the upper bound of the blow-up rate of blow-up solutions to (1)-(3), we need an extra assumption on initial data $u_0$ as follows

$$(H) \quad \{ u(t_0, \cdot), (u^n(t_0, \cdot)) \geq 0 \}.$$

Remark 1.2. We can easily obtain from the assumption that $u(t) \geq 0$ for $t \in (0,T)$ (see [8, 33, 34]).

Theorem 1.5. Suppose that the initial value $\mathbf{u}_0(\cdot, \cdot)$ satisfies $(H)$. If the solution $u(x,t)$ of (1)-(3) blows up in finite time $T$, then there exists a positive constant $C$ such that

$$u(t,x) \leq C(T-t)^{-\frac{p+1}{(p+1)(p+2m+1)}}$$

The rest of this paper is organized as follows. In Section 2, we study the decay behavior of the solution and establish critical exponents of the problem (1)-(3). In Section 3 we consider the critical case $q = q_c$ and prove Theorem 1.3. The blow-up set and the estimate of the blow-up rate are considered in Section 4.

II. DECAY BEHAVIOR AND CRITICAL EXPONENTS

In this section, we begin with the decay behavior of the solution to (1)-(3), which plays an important role in the proofs of Theorems 1.2-1.3.

Proof of Theorem 1.1. Our idea is to show that any positive solution of the problem (1)-(3) is, for $x$ large, bigger than the following similarity solution

$$U_\lambda(t,x) = \lambda^{\frac{p+1}{m}} \mathcal{U}_1(t,\lambda x),$$

where

$$U_\lambda(t,x) = \left\{ \begin{array}{ll} \frac{p+1}{2p+1}\left[ 1 + C_{m,p} \right] & \frac{p+1}{2p+1} \left( \frac{p+1}{p+2} \right) \frac{p+1}{p+2m+1} \\
0 & \lambda > 0 \end{array} \right.$$ 

Let $0 < \tau < T$ and $S = [\tau, T] \times [0,\infty)$. Since the positive solution $u(t,x) = \min u(x,t)$ is continuous in $(0,T] \times [0,\infty)$, there exists $\delta = \min u(x,t), \tau \leq t \leq T, 0 \leq x \leq 1$ such that

$$\delta = \min u(x,t), \tau \leq t \leq T, 0 \leq x \leq 1. (9)$$

We now select $\gamma > 0$ such that

$$U_\lambda(t-x, \tau) \leq \delta, \tau \leq t \leq T, x \geq \frac{1}{2}. (10)$$

To this aim, according to the definition of $U_\lambda(t,x)$ we need
We have shown such that \( C_{v} \). The proof of Theorem 1.1 is completed.

On the other hand, on the boundary we have
\[
\delta = \lambda \frac{p+1}{p} \left( t - \tau \right) \left( 1 - \frac{m}{m+1} \right) + C_{v} \left( 1 - \frac{m}{m+1} \right) \left( t - \tau \right) \frac{1}{p+1},
\]
for \( \tau \leq t \leq T \), and \( x \geq \frac{1}{2} \), which is implied by
\[
\delta = \lambda \frac{p+1}{p} \left( t - \tau \right) \left( 1 - \frac{m}{m+1} \right) + C_{v} \left( 1 - \frac{m}{m+1} \right) \left( t - \tau \right) \frac{1}{p+1}, \tag{11}
\]
Since the right-hand side of (11) is bounded below by \( \frac{\lambda}{p+1} \), where \( \lambda = \rho(m, p) > 0 \), the inequality (11) is satisfied if we choose \( \lambda \) such that \( \lambda \leq \rho(m, p) \). Since \( \partial U_{\lambda} / \partial t = \delta \partial U_{\lambda} / \partial x \) in \( S \) and \( U_{\lambda}(t - \tau) = 0 \) for \( t = \tau, x \geq 1 \), by (9), (10) and the comparison principle we have
\[
U_{\lambda}(t - \tau, x) \leq U(t, \tau), \quad \tau \leq t < T, \quad x \geq 1.
\]
Hence
\[
\liminf_{x \to +\infty \text{ or } x \to -\infty} x^p \lim_{x \to +\infty} U(t - \tau, x) = \liminf_{x \to +\infty \text{ or } x \to -\infty} x^p \lim_{x \to +\infty} U_{\lambda}(t - \tau, x) \leq \liminf_{x \to +\infty \text{ or } x \to -\infty} x^p \lim_{x \to +\infty} U(t, \tau), \tag{12}
\]
Since the right-hand side of (12) does not depend on \( \lambda \), the estimate (9) holds by letting \( \lambda \) tend to zero and \( T \) tend to \( T \). The proof of Theorem 1.1 is completed.

The rest part of this section is devoted to discussion of the critical exponents and prove Theorem 1.2. First, we show critical global existence exponent, and characterize when all solutions to the problem (1)-(3) are global in time or they blow up.

Lemma 2.1. If \( 0 < q \leq q_{0} = \frac{2p + m + 1}{p + 2} \), every nontrivial and nonnegative solution exists globally in time.

Proof. In order to prove that the solution \( u \) of (1)-(3) is global, we look for a globally defined in time supersolution of the self-similar form
\[
\Pi(x, t) = e^{\gamma t}(M + e^{-2\gamma t}x)^{\frac{1}{m}}, \quad x \geq 0, t \geq 0,
\]
where
\[
M = \max \left\{ \left| u_{0} \right|^{\frac{1}{m}}, 1 - \frac{p - m}{p + 2}M \right\}, \quad L = m^{p+1}M(1 + M)^{\frac{1}{m-1}},
\]
\[
\kappa_{1} = \frac{1}{m} \left( 1 - \frac{m}{m+1} \right), \quad \kappa_{2} = \frac{1}{m} \left( 1 - \frac{p - m}{p + 2}M \right).
\]
Obviously, we have
\[
\Pi(x, 0) \geq u_{0}(x), \quad \Pi(x, 0) \geq v_{0}(x), \quad x \geq 0. \quad \text{Since } -ye^{-\gamma t} \geq -e^{t} \text{ for } y > 0, \text{ after a direct computation, we obtain}
\]
\[
\Pi = \kappa_{1} e^{\gamma t} \left( M + e^{-2\gamma t}x \right)^{\frac{1}{m}} - \kappa_{2} L e^{\gamma t} e^{-2\gamma t} \left( M + e^{-2\gamma t}x \right)^{\frac{1}{m-1}},
\]
\[
\geq \kappa_{1} e^{\gamma t} \left( M + e^{-2\gamma t}x \right)^{\frac{1}{m}} - \kappa_{2} e^{-\gamma t} \left( M + e^{-2\gamma t}x \right)^{\frac{1}{m}},
\]
\[
\geq \left( 1 - \frac{p - m}{p + 2}M \right) \left( 1 - \frac{m}{m+1} \right) \left( e^{\gamma t} \right)^{\frac{1}{m}},
\]
\[
\Pi(x, 0) = (p + 1)^{\frac{1}{m}} - \Pi(x, 0) = (p + 1)^{\frac{1}{m}} - \Pi(x, 0), \tag{13}
\]
In \( R_{+} \times R_{+} \). On the other hand, on the boundary we have
\[
-\left| \Pi_{\gamma} \right|^{\gamma}(\Pi^{\gamma}), (0, t) = \frac{L^{p+1}}{m^{p+1}} e^{\gamma t}(k + \kappa_{1}) \gamma \left( 1 - \left( 1 - \frac{p - m}{p + 2}M \right)^{\gamma} \right) \left( M + \gamma \left( 1 - \frac{p - m}{p + 2}M \right)^{\gamma} \right)^{\gamma},
\]
\[
\gamma \left( 1 - \left( 1 - \frac{p - m}{p + 2}M \right)^{\gamma} \right) \left( M + \gamma \left( 1 - \frac{p - m}{p + 2}M \right)^{\gamma} \right)^{\gamma}, \tag{14}
\]
By the definition of \( \kappa_{1}, \kappa_{2}, L, M \) and the assumption
\[
0 < q \leq \frac{2p + m + 1}{p + 2},
\]
we can check that
\[
\Pi(x, t) \geq \Pi(0, t) \geq \Pi(x, 0), \text{ for } t > 0. \quad \text{We have shown that } \Pi \text{ is a supersolution of the problem (1)-(3), which implies that every solution of the problem (1)-(3) is global provided } 0 < q \leq \frac{2p + m + 1}{p + 2} \text{ by the comparison principle.}
\]

Lemma 2.2. If \( q > q_{0} = \frac{2p + m + 1}{p + 2} \), then the solution of the problem (1)-(3) with appropriately large initial data blows up in finite time.

Proof. To prove the non-existence of global solutions, we construct a blow-up self-similar subsolution of the system. Construct
\[
\mathbf{u}(x, t) = (T - t)^{-\frac{1}{2}} f(\xi), \quad \xi = x(T - t)^{-\frac{1}{2}} \tag{13}
\]
where \( T \) is a positive constant, \( f \) is to be determined later and
\[
k = \frac{p + 1}{(p + 2)q - (2p + m + 1)}, \quad l = \frac{q - m - p}{q(p + 2) - (2p + m + 1)}. \tag{14}
\]
After some computations, we have
\[
\mathbf{u}_{\gamma} = (T - t)^{-(m - 1)}(k f(\xi) + l f'(\xi)),
\]
\[
\left| \mathbf{u}_{\gamma} \right|^{\gamma}(u^{\gamma}), (0, t) = (T - t)^{p - m + 1 - k} f'(\xi)(\xi), \tag{15}
\]
\[
\left| \mathbf{u}_{\gamma} \right|^{\gamma}(u^{\gamma}), (0, t) = (T - t)^{p - m + 1 - k} f'(\xi)(\xi), \tag{16}
\]
\[
| \mathbf{u}_{\gamma} |^{\gamma} | u^{\gamma} |(0, t) = (T - t)^{p - m + 1 - k} f'(\xi)(\xi), \tag{17}
\]
\[
| \mathbf{u}_{\gamma} |^{\gamma} | u^{\gamma} |(0, t) = (T - t)^{p - m + 1 - k} f'(\xi)(\xi), \tag{18}
\]
\[
(0, t) = (T - t)^{-\frac{1}{2}} f'(\xi), \tag{19}
\]
Thus, \( (u, v) \) is weak subsolution of (1)-(3) provided that
(\| f'^1 \| (f'^m)'(t^\xi)) \geq k\xi f^1(\xi) + f'(\xi)\xi, \quad (15) \]

\[- | f'^1 | (f'^m)'(0) \leq f^1(0). \quad (16) \]

Set

\[ f(\xi) = (A + B\xi) - \frac{p+2}{1-p-m} \tag{17} \]

where \( A, B \) are positive constants to be determined. It is easy to see that

\[ f'(\xi) = -B \frac{p+2}{1-p-m} (A + B\xi)^{-\frac{p+2}{1-p-m}}, \quad (18) \]

\[ | f'^1 | (f'^m)' = -mB^{\frac{1}{p+2}} \left( \frac{p+2}{1-p-m} \right) \frac{p+2}{1-p-m} (A + B\xi)^{-\frac{p+2}{1-p-m}}. \quad (19) \]

\[ \langle f'^1 \rangle (f'^m)' \rangle = \frac{2p+m+1}{1-p-m} (A + B\xi)^{-\frac{p+2}{1-p-m}}. \quad (20) \]

Substituting (17)-(20) into (15), then inequality (15) is valid provided that

\[ k(A + B\xi)^{-\frac{p+2}{1-p-m}} - \frac{2p+m+1}{1-p-m} (A + B\xi)^{-\frac{p+2}{1-p-m}} \leq 0. \]

By taking

\[ B \geq \left[ \frac{2p+m+1}{1-p-m} \left( \frac{1-p-m}{p+2} \right)^{\frac{1}{p+2}} \right]^\frac{1}{p+2} \]

and

\[ A \leq \left( \frac{2p+m+1}{1-p-m} \right)^{\frac{1}{p+2}} \left( \frac{p+2}{1-p-m} \right)^{1/(p+2)} \]

it is easy to check that (15) and (16) are valid. Noticing that

\[ q > q_0 = \frac{2p+m+1}{p+2} \]

implies that \( k > 0, \) \( u \) given by (15) and (17) is a blowup subsolution of the problem (1)-(3) with appropriately large \( t_0. \) It follows from the comparison principle that the problem (1)-(3) exists a solution blowing up in a finite time.

Now we turn our attention to the critical exponent of Fujita type. That is, we shall show when all solutions of (1)-(3) blow up in a finite time or both global and nonglobal solutions exist.

Lemma 2.3. If \[ \frac{2p+m+1}{p+2} = q_0 < q < q_c = \frac{2p+m+1}{p+2}, \]

then each positive solution of the problem (1)-(3) blows up in finite time.

Proof. Without loss of generality, we first assume that \( u \) is nonincreasing in \( x \), for if not we consider the (nonincreasing in \( x \) ) solution \( \omega \) corresponding to the initial value \( \omega_b(x) = \inf \{ u_b(y), 0 \leq y \leq x \} \), which are nonincreasing in \( x \). If \( \omega \) blows up in finite time, so does \( u \). On the other hand, for every \( \epsilon > 0 \) and \( t_0 > 0 \) fixed, by Theorem 1.1, there exists a constant \( M > 0 \) large enough that

\[ u(x,t_0) \geq \left( \frac{C_{a,p} \epsilon + x}{t_0^{1/(p+1)}} \right)^{\frac{1}{1-m-p}}, \quad x \geq M, \]

and

\[ u(x,t_0) \geq u(M,t_0), \quad 0 \leq x \leq M. \]

Now we construct the following well-known self-similar solution (the so-called Zel'dovich-Kompaneetz-Barenblatt profile [8, 12, 24]) to (1)-(3) in the form

\[ u_b(x,t) = (\tau + t)^{-\frac{1}{m+1-p+1}} h(\tau + t), \quad x = \tau + t \]

where \( \tau > 0, b > 0 \) and \( C_{a,p} \) is given in (10). It is not difficult to check that

\[ \langle f'^1 \rangle (h'^m)' \rangle + \frac{1}{m+2+p+1} h'(h'^m) \]

with \( \tau > 0, b > 0 \) and \( C_{a,p} \) is given in (10). It is not difficult to check that

\[ \langle f'^1 \rangle (h'^m)' \rangle + \frac{1}{m+2+p+1} h'(h'^m) \]

Then each positive solution of the problem (1)-(3) blows up in finite time. If \$\omega\$ blows up in finite time, so does \$u\$. On the other hand, for every \$\epsilon > 0 \$ and \$t_0 > 0 \$ fixed, by Theorem 1.1, there exists a constant \$M > 0 \$ large enough that

\[ u(x,t_0) \geq \left( \frac{C_{a,p} \epsilon + x}{t_0^{1/(p+1)}} \right)^{\frac{1}{1-m-p}}, \quad x \geq M, \]

and

\[ u(x,t_0) \geq u(M,t_0), \quad 0 \leq x \leq M. \]

We declare that there exist \$t_0 \geq t_0 \$ and \$T \$ large enough that

\[ u_b(x,t) \geq u(x,0), \quad x \in (0, +\infty), \] \quad (23) \]

where \$u(x,t) \$ is given by (13) and (17). A simple calculation shows that (23) is valid provided

\[ (\tau + t)^{-\frac{1}{m+1-p+1}} \gg T^{-k}, \quad (24) \]

\[ (\tau + t)^{-\frac{1}{m+1-p+1}} \ll T^{-k}. \quad (25) \]

Since \$q_0 < q < q_c \$, we have \$k > 1 \$. Thus there exist \$t_0 \geq t_0 \$ and \$T \$ large enough that (24) and (25) are both valid. Thus \$u(x,t) \geq u_b(x,t) \geq u(x,0), \quad x \in (0, +\infty), \$ which with the comparison principle implies that \$u \$ blows up in a finite time.
Lemma 2.4. If $q > q_c$, then the problem (1)-(1) admits positive global solutions with small initial data.

Proof. We investigate the auxiliary function
\[ \bar{u}(x,t) = (\tau + t)^{-k} F(\xi), \quad \xi = x(\tau + t)^{-k}, \]
where $\tau$ is a positive constant, $F(\xi)$ is to be determined later and
\[ k = \frac{p+1}{(p+2)q - 2(p+m+1)q}, \quad l = \frac{q-m-p}{q(p+2) - 2(p+m+1)}. \]  
By a direct computation, we obtain
\[ \bar{u}_t = (\tau + t)^{-k+1}(-k F'(\xi) - l^2 F''(\xi)), \]
\[ \int \bar{u}_t (\bar{u}^m) = (\tau + t)^{-k+1} \int F''(F^m)(\xi), \]
\[ \int \bar{u} \bar{u}_t (\bar{u}^m) = (\tau + t)^{-k+1} \int F''(F^m)(\xi), \]
and
\[ \int \bar{u}_t (\bar{u}^m)(0,t) = (\tau + t)^{-k+1} \int F''(F^m)(0), \]
\[ \bar{u}(0,t) = (\tau + t)^{-k+1} F^m(0). \]
Thus, $\bar{u}$ is weak supsolution of (1)-(3) provided that
\[ \| F''(F^m)(\xi) \| + kF(\xi) + lF'(\xi) \xi \leq 0, \]
(28)

We choose
\[ F(\xi) = H \left( (ab)^{p+1} + (\xi + a)^{p+1} \right)^{\frac{p+1}{1-p-m}} \]
with $b > 0, H > 0, a > 0$ to be determined. After a series computations, we obtain
\[ F''(\xi) = -H \frac{p+2}{1-p-m} \left( (ab)^{p+1} + (\xi + a)^{p+1} \right)^{\frac{p+1}{1-p-m}} (\xi + a)^{\frac{1}{p+2} - 1}, \]
\[ \| F''(F^m) \| = mH^{p+1} \frac{p+2}{1-p-m} \left( (ab)^{p+1} + (\xi + a)^{p+1} \right)^{\frac{p+1}{1-p-m}} (\xi + a), \]
\[ \| F''(F^m) \| = mH^{p+1} \frac{p+2}{1-p-m} \left( (ab)^{p+1} + (\xi + a)^{p+1} \right)^{\frac{p+1}{1-p-m}} (\xi + a)^{\frac{1}{p+2} - 1}, \]
substrating above equalities into (28), let $y = \xi + a$, then (28) can be transformed into the following inequality with respect $y$
\[ G(y) = -e_1 y^{p+1} + e_2 a y^{p+1} - e_3 (ab)^{p+1} \leq 0, \]
(31)
where
\[ e_1 = mH^{p+1} \left( \frac{p+2}{1-p-m} \right)^{\frac{1}{p+1}} - Hk + lH \frac{p+2}{1-p-m} - mH^{p+1} \left( \frac{p+2}{1-p-m} \right)^{\frac{1}{p+1}}, \]
\[ e_2 = lH \frac{p+2}{1-p-m} - Hk, \]
\[ e_3 = mH^{p+1} \left( \frac{p+2}{1-p-m} \right)^{\frac{1}{p+1}}. \]
Since the assumptions $q > q_c = 2p + m + 1$ imply $l > k > 0$ we can choose a suitable constant $H > 0$ such that
\[ l > mH^{p+1} \left( \frac{p+2}{1-p-m} \right)^{\frac{1}{p+1}} > k > 0, \] for such $H$, it is easy to verify that $e_1 > 0 (i = 2, 3)$ and $G(y)$ is a concave function with respect to $y^{p+1}$, then $G(y)$ attains its maximum at $y = -\frac{e_2 d}{(p+2)e_1}$. Therefore, (31) is valid provided that
\[ G(y) = a^{\frac{p+1}{p+2}} \left( \frac{p+1}{p+2} \right) (\frac{e_2}{(p+2)e_1})^{p+1} \leq 0. \]
(32)

So, we only need to choose $b$ sufficiently large such that
\[ b \geq \left( \frac{p+1}{p+2} \right) (\frac{e_2}{(p+2)e_1})^{\frac{p+1}{p+2}}. \]
Consequently, we have proved that inequalities (28) is true.

Now we consider the boundary condition (29), for above fixed $H, d$, we only need to show that
\[ a^{\frac{p+1}{p+2}} \left( \frac{p+1}{p+2} \right) \left( \frac{e_2}{(p+2)e_1} \right)^{\frac{p+1}{p+2}} \leq 0. \]
(33)

The assumption $q > 2p + m + 1$ ensures that there exist $a$ large enough such that the inequality (33) holds, which implies that the inequality (29) holds.

Thus, for the case $q > q_c = 2p + m + 1$, we have constructed a class of global selfsimilar supsolutions defined by (26) and (30). Owing to the comparison principle, the solution of the problem (1)-(3) is global if the initial data $u_0$ is small enough.

III. THE CRITICAL CASE $q = q_c$

In this section, by the stationary state technique used in [8, 11], we study the critical case $q = q_c$.

Proof of Theorem 1.3. Assume by contradiction that there exists a global nontrivial solution $u(x,t)$. Let $v(x,t) = u(x,1+t)$. Then $v(x,t)$ is the solution of the problem (1)-(1) with initial data $v_0(x) = u(x,1)$. Using the spatial decay given in (9), we can choose $a > 0$ and $b > 0$ such that $v_0(x) \geq g(x+b)$, where $g$ is the Barenblatt profile given in (22). Now we make the following change of variables
\[ h(\xi,\tau) = (1+t)^{-\frac{1}{p+1}+1} \left( \frac{1}{p+1} + 1 \right)^{\frac{p+1}{p+1}+1} \left( \frac{1}{p+1} + 1 \right)^{\frac{p+1}{p+1}+1}, \]
where $\tau = \log(1+t)$ denotes the new time. Then it is easily checked that $h(\xi,\tau)$ is the solution of the following problem.
Let \( h(\xi, \tau) \) be the corresponding solution with initial data \( g(\xi + b) \). It follows that \( h \geq h \), therefore \( h \) is also global.

It can be easily verified that
\[
\| g \|_{L^p(\mathbb{R})}^p (\xi + b) + \frac{1}{m+2p+1} h(\xi) + H(\xi) = 0.
\]

Hence we can show that \( h \) is nondecreasing in \( \tau \) (see [8]), moreover, we know that \( h \) is nondecreasing in \( \xi \).

Next we will prove that for any \( \tau > 0 \) we have
\[
\lim_{\tau \to \infty} h(\xi, \tau) = H(\xi) = 0.
\]

Otherwise,
\[
\lim_{\tau \to \infty} h(\xi, \tau) = +\infty
\]
uniformly on \([0, \xi_0]\), since \( h \) is nonincreasing in \( \xi \) in \( \mathbb{R} \).

Therefore we claim that \( v(x,t) \) blows up in finite time (the claim is to be proved later). However, \( v \) was assumed to be global. This contradiction shows that the function \( H(\xi) \) is well defined. In view of the regularity of the bounded solution of the problem (3.1), by using the standard arguments [8], we can pass to the limit in the first equation of (34) to get.

\[
\| H' \|^2 + H = 0.
\]

Because of the regularity of \( h \) in the region where \( H > 0 \) [8], we can pass to the limit in the boundary condition in (34) for \( -|H'|^2 H''(0, \tau) = H_{2m+1}^2(0, \tau) \), and obtain \( -|H'|^2 H''(H(m)) = 0 \) whenever \( s > 0 \). However, the every solution of (36) is a Barenblatt profile and this profile does not satisfy the boundary condition, leading to contradiction. The proof of Theorem 1.3 is completed.

Now we prove the claim. By (35) there is positive \( \tau_0 \) such that \( h(\xi, \tau_0) > N \) on \([0, \xi_0] \) for any \( N > 0 \). In other words, at time \( t_0 = \tau_0 - 1 \), the profile \( h(\xi, \xi) \) in the original variables satisfies \( v(x, t_0) \geq (1 + t_0)^{-1} \), by Theorem 1.1, there exists a constant \( t_0 \) large enough that
\[
v(x, t_0) = u(x, 1 + t_0) \geq C_1(1 + t_0)^{-1} x^{-2m-1}.
\]

for \( x > h(1 + t_0)^{-1} \), and
\[
\lim_{\tau \to \infty} h(\xi, \tau) = +\infty
\]
uniquely on \([0, \xi_0]\), since \( h \) is nonincreasing in \( \xi \) in \( \mathbb{R} \).

Therefore we claim that \( v(x,t) \) blows up in finite time (the claim is to be proved later). However, \( v \) was assumed to be global. This contradiction shows that the function \( H(\xi) \) is well defined. In view of the regularity of the bounded solution of the problem (3.1), by using the standard arguments [8], we can pass to the limit in the first equation of (34) to get.

\[
\| H' \|^2 + H = 0.
\]

Because of the regularity of \( h \) in the region where \( H > 0 \) [8], we can pass to the limit in the boundary condition in (34) for \( -|H'|^2 H''(0, \tau) = H_{2m+1}^2(0, \tau) \), and obtain \( -|H'|^2 H''(H(m)) = 0 \) whenever \( s > 0 \). However, the every solution of (36) is a Barenblatt profile and this profile does not satisfy the boundary condition, leading to contradiction. The proof of Theorem 1.3 is completed.

Now we prove the claim. By (35) there is positive \( \tau_0 \) such that \( h(\xi, \tau_0) > N \) on \([0, \xi_0] \) for any \( N > 0 \). In other words, at time \( t_0 = \tau_0 - 1 \), the profile \( h(\xi, \xi) \) in the original variables satisfies \( v(x, t_0) \geq (1 + t_0)^{-1} \), by Theorem 1.1, there exists a constant \( t_0 \) large enough that
\[
v(x, t_0) = u(x, 1 + t_0) \geq C_1(1 + t_0)^{-1} x^{-2m-1}.
\]

for \( x > h(1 + t_0)^{-1} \), and
\[
\lim_{\tau \to \infty} h(\xi, \tau) = +\infty
\]
\[ \omega_j = \left\{ \omega_j \right\}_{j=1}^n (\omega_n^+) \cap (0, x < R, 0 < t < T^*), \]
\[ \omega(0, t) = s(T^*), \quad t \in (0, T^*), \]
\[ \omega(x, 0) = u_0(x), \quad 0 < x < R. \]

Construct the supersolution

\[ U_j(x, t) = (t + 2)^{1-m/r} \eta_j(x), \]
where \( \eta_j \) is a solution of the elliptic problem (see [1, 20])

\[ \left( |\eta_j'|^m (\eta_j^n)^m \right) - \frac{1}{1-p-m} \eta_j = 0. \]

\[ \eta_j(0) = \eta_j(R) = \infty, \]

where \( R \) and \( r \) are large enough that \( R > x_0, U_j(x, 0) = u_0(x) \) for \( 0 < x < R \). By using the comparison principle, we have that

\[ U_j(x, t) \geq \omega(x, t) \geq u(x, t), \quad (x, t) \in (0, R) \times (0, T^*). \]

For the arbitrariness of \( T^* \), we have

\[ U_j(x, t) \geq w(x, t), \quad (x, t) \in (0, R) \times (0, T^*), \]

which leads to a contradiction. The proof of Theorem 1.4 is completed.

For the rest of the section we need to assume that initial data satisfies \( H \). As a consequence we have that

\[ (|u_{tt}|^m (u_n^n)_t) \geq 0. \]

Clearly, \( u(0, t) = \max_{u(0, 0)} u(x, t) \).

Let

\[ M(t) = u(0, t), \quad a = \frac{m-r}{p-1}, \quad b = \frac{2p+1-q(p+1)}{p+1}, \]

and define the function \( \psi_M(y, s) = \frac{1}{M(t)} u(ay, bs + t) \) in

\[ R_+ \times \left( \frac{t}{b} \right) \] for \( t < T^* \). Then \( \psi_M \) satisfies

\[ 0 \leq \psi_M \leq 1, \quad \psi_M = 1, \quad (\psi_M)_t \geq 0. \]

Moreover, \( \psi_M \) is a solution of the following problem

\[ \left( (\psi_M)_t \right) = \left( \left( (\psi_M^n)_t \right) \right), \quad x \geq 0, \quad -\frac{t}{b} < s < 0, \]

\[ -\left( (\psi_M)_t \right) \left( (\psi_M^n)_t \right), (0, s) = (\psi_M)^n(0, s), -\frac{t}{b} < s < 0. \]

The following lemma is basic for the blow-up estimate. Lemma 4.1. Under the assumptions of Theorem 1.5, there exists constant \( c \) for large enough such that

\[ (\psi_M^\prime)(0, 0) \geq c. \]

Proof. We will prove \( (\psi_M^\prime)(0, 0) \geq c \). Otherwise, e.g., there exists a sequence \( M_j \rightarrow \infty \) such that

\[ \frac{\partial \psi_M}{\partial s}(0, 0) \rightarrow 0. \]

Since \( \psi_M \) is uniformly bounded in \( C^{2, \frac{\alpha}{2}} \), passing to a subsequence we have \( \psi_M \rightarrow \psi \) for some positive function \( \psi \) in \( C^{2, \frac{\beta}{2}}(\beta < \alpha) \) satisfying

\[ 0 \leq \psi \leq 1, \psi(0, 0) = 1, \frac{\partial \psi}{\partial s} \geq 0, \]

which is a weak solution of

\[ (\psi^\prime), (\psi^n), x \geq 0, \quad s, s < 0, \]

\[ -\left( (\psi^\prime) \right) \left( (\psi^n)_t \right), (0, s) = (\psi^n)(0, s), s, s < 0, \]

where \( s \) is some constant satisfying \( -\frac{t}{b} < s \). Set \( w = \psi \). Then \( w \) satisfies

\[ w_t = (p + m)(\psi^\prime) \left( \psi^n \psi^{n-1} \right), x \geq 0, \quad s, s < 0, \]

\[ -\left( (\psi^\prime) \right) \left( \psi^n \psi^{n-1} \psi \right), (0, s) = q \psi^n(0, s), s, s < 0. \]

Thus \( w \) has minimum at \( (0,0) \) with \( w(0, 0) = 0 \). By using Hopf’s lemma we know that \$w \equiv 0\$, which means that \( \psi \) does not depend on \( s \). Thus

\[ 0 = (\psi)_t = \left( (\psi^\prime) \right) \left( \psi^n \right), \quad (y, s) \geq 0, \quad s < 0, \]

\[ -\left( (\psi^\prime) \right) \left( (\psi^n) \right), (0, s) = (\psi^n)(0, 0) = 1, \]

\( \psi \) is unbounded. This contradiction proves the lemma.

Now we give the proof for Theorem 1.5 as follows. Proof of Theorem 1.5. If we rewrite the inequality (39) in terms of \( M(t) \), we obtain

\[ M(t) \leq \left( \frac{p+1}{p-m-q(p+1)} \right) (t)M'(t). \]

Integrating and taking into account that \( M(t) = u(0, t) \), we get

\[ u(., t) \leq C(T - t) \]

The proof of Theorem 1.5 is completed.

REFERENCES


[36] Jin Li was born in Gansu on January, 1957. Associate professor. The main research direction is Nonlinear partial differential equations and Applied analysis.