Impulsive Tracking Control for Non-measurable State with Time-delay

Yuanqiang Chen  
College, Guizhou Minzu University, Guiyang550025, China  
Email: yuanqiango@126.com

Renbi Tian  
Department of Education Science, Tongren University, Tongren554300, Chain  
Email: 28906406@qq.com

Abstract—In this paper, we discuss the problem by utilizing impulsive control and Lyapunov function methods, which is about the state of disturbed systems with time-delay tracking the state of reference systems. Sufficient conditions for the solvability of the tracking control problem are given for the measurable state and the non-measurable state of system respectively. This impulsive control law based on measured output instead of the state information is considered. Finally, a numerical example is presented to illustrate the validity of our results.

Index Terms—Impulsive control, State tracking, Non-measurable state

I. INTRODUCTION

The research of the tracking control without time-delay [1]-[3] is quite mature. In many evolutionary systems there are two common phenomena: delay effects and impulsive effects. In implementation of electronic system, for example, delays frequently appear because of the finite switching speed of amplifiers. On the other hand, the state of electronic system is often subject to instantaneous perturbations and experience abrupt change at certain instants which may be caused by switching phenomenon, frequency change or other sudden noise, that is, do exhibit impulsive effects. Even in biological system, impulsive effects are likely to exist. For instance, when a stimulus from the body or the external environment is received by receptors the electrical impulses will be conveyed to the neural net and impulsive effects arise naturally in the net. Therefore, the tracking control system with delays and impulsive effects should be more accurate to describe the evolutionary process of the systems. Since delays and impulses can affect the dynamical behaviors of the system by creating oscillatory and unstable characteristics, it is necessary to investigate both delay and impulsive effects on the stability of tracking control system. The delaying time, which is the tracking signal’s transmission and influence to realizing of tracking performance, is very big. Therefore, to consider the tracking problem with time-delay is more significance, it has already been researched recently, for example [4]-[5]. In fact, the tracking system is disturbed by external environment. So the tracking control problem, which contains the delaying time and disturbance of the external environment, is more comprehensive. Impulsive control method [6]-[26] has attracted considerable attention because impulsive control laws have fast response time, low energy consumption, good robustness and resistance to disturbances. But studies of the tracking control, specially, the tracking control with time-delay and disturbance, by the impulsive control is quite rare now.

In this paper, we give a reference linear system

\[ \dot{x}_r(t) = A_r x_r(t), \]

\[ x_r(0) = x_{r0}, \] (1)

And a tracking linear system with time-delay and disturbance

\[ \dot{x}(t) = A x(t) + B x(t-h_{\sigma}(t)) + w(t), \]

\[ y(t) = C x(t), \]

\[ x(t) = \phi(t), t \in [-h, 0]. \] (2)

And the state tracking performance index

\[ \lim_{t \to \infty} \| x(t) - x(t) \| = 0, \] (3)

where \( x_r(t), x(t) \in R^n \) is the state variable of the reference system and tracking system respectively, \( y(t) \in R^m \) is output, \( A, A_B, \) and \( C \) are constant matrices of appropriate dimensions, \( w(t) \) is bounded external disturbance which is the continuous vector valued function, \( h_{\sigma}(t) \) is the delay time, and \( h_{\sigma}(t) \leq h_{\sigma}(t) \leq h_{\sigma}(t) \leq h_{\sigma}(t) \leq h_{\sigma}(t) \).

We will adopt the impulsive control method and the Lyapunov functional method and the matrix inequality technology to solve the state tracking system with time-delay and disturbance for the cases that the state of system is non-measurable, and give the sufficient conditions for the realizing of the tracking control performance. The rest of this paper is organized as...
follows. In Section II, some preliminary lemmas are presented. In Section III, based on the impulsive control method and the Lyapunov functional method and the matrix inequality technology, sufficient conditions for the solvability of the tracking control problem are given for the measurable state and the non-measurable state of system respectively. Moreover, a numerical example is presented in Sections IV. Section V concludes the paper.

II. PRELIMINARIES

In this paper, $P > 0 (\succeq, \preceq, \prec)$ denotes a positive definite (semi-definite, negative definite, semi-negative definite) matrices $P$, $\lambda_{\text{max}}(P)$ and $\lambda_{\text{min}}(P)$ are respectively the largest and the smallest eigenvalue of $P$, $\|\cdot\|$ denotes the norm in $R^n$, and $K$ denotes the set of continuous functions. $PC(R, R)$ is the set of all piecewise continuous functions $p: R \rightarrow R$, such that $p \in PC(R, R)$, if $p$ where, $p: R \rightarrow R$, is continuous on $R$, except at the time points in the set $\{t_k\}$, and is left-continuous and has right limit at $t_k$ for all $k$.

We first introduce some preliminary concepts which will be found useful in the paper. Consider the following impulsive control system with time-delay:

$$\dot{x}(t) = f(t, x(t), x(t-h)), t \neq t_k,$$

$$\Delta x(t) = x(t^+) - x(t^-) = u_k(x), t = t_k, \quad (4)$$

where $f, u_k \in C(R \times R^n, R^n)$, while $0 < t_1 < \cdots < t_k < \cdots$, with $t_k \rightarrow \infty$ as $k \rightarrow \infty$.

Definition 1. For each $\rho > 0$, define

$$S_{\rho} = \{x \in R^n : \|x\| < \rho\},$$

And for $(t, x) \in \{(t_{k-1}, t_k) \times R^n, k = 1, 2, \ldots, \}$

$$D^rV(t, x) = \limsup_{k \rightarrow \infty} - \frac{1}{h} [V(t, x) - V(t - h, x)]$$

Definition 2. Let $V_0$ be the set containing all functions $V(\cdot) : R \times S_{\rho} \rightarrow R$, which are continuous on $R \times S_{\rho}$, except possibly at a sequence $\{t_k\}$ of points, and satisfy the following two conditions:

i) For each $x \in S_{\rho}, k = 1, 2, \ldots, \lim_{(t, x) \rightarrow (t_k, x)} V(t, x) = V(t_k, x)$ exists;

ii) $V(t, x)$ is locally Lipschitz in $x$.

The following lemma gives sufficient conditions for asymptotic stability of system (4).

Lemma 1.[27] Assume that there exist $\alpha, \beta, \gamma, g \in K$, $p \in PC(R, R)$, $V(\cdot) \in V_0$ and $\sigma > 0$, such that the following conditions are satisfied.

i) $\beta(\|x\|) \leq V(t, x) \leq \alpha(\|x\|), \forall \langle t, x \rangle \in [-h, \infty) \times S_{\rho};$

ii) $V\left(t_k, \varphi(0) + u_k \varphi(t_k)\right) \leq g\left(V(t_k, \varphi(0))\right), \forall \langle t_k, \varphi \rangle \in R \times PC([-h, 0], S_{\rho});$

iii) $D^rV\langle t, \varphi(0) \rangle \leq \rho(t) \gamma(V\langle t, \varphi(0) \rangle), \forall t \in R, t \neq t_k$ an $d\varphi \in PC([-h, 0], S_{\rho}),$ when

$$V\langle t, \varphi(0) \rangle \geq g\left(V\langle t + s, \varphi(s) \rangle\right), \forall s \in [-h, 0);$$

iv) $G_2 = \inf_{\tilde{\gamma} > 0} \int_{0}^{\infty} \frac{ds}{\tilde{\gamma}(s)} > \sup_{p > 0} \int_{0}^{\infty} p(s) ds = G_1$

where $\varphi(\tau) = \varphi(0)$, $\tau = \sup\{\tau_k - t_k\} < \infty$.

Then, the system (4) is asymptotically stable.

Lemma 2.[28] Let $M, N$ be real matrices of appropriate dimensions. Then, for any matrix $S > 0$ of appropriate dimension and any scalar $\gamma > 0$, the following inequality holds.

$$MN + N^T M^T \leq \gamma^{-1} MS^{-1} M^T + \gamma N^T N.$$
For the following tracking system, each state variable of which has not same delay time.

\[ \begin{align*}
\dot{x}(t) &= Ax(t) + Bx(t-h_n(t)) + w(t), \\
y(t) &= Cx(t), \\
x(t) &= \phi(t), t \in [-h, 0],
\end{align*} \]

Where

\[ x(t-h_n(t)) = (x_1(t-h_n(t)), x_2(t-h_n(t)), \ldots, x_n(t-h_n(t)))^T. \]

We have the following corollaries from Definition 1 and Theorem 1.

**Corollary 1.** Let \( h_i(t) \leq \tilde{h}_i (i = 1, 2, \ldots) \), under the impulsive control law \( \{\tau_i, Q Q x, (t) - Q x(t)\} \), the state of system (8) tracks asymptotically and impulsively that of the reference system (1) if the system controlled satisfies the conditions of Theorem 1.

**Proof.** Consider the following Lyapunov function candidate

\[ V(t, x) = x(t)^T P x(t). \]

It clearly satisfies condition (1) of Lemma 1. At \( t = \tau_i \), \( i = 1, 2, \ldots \), we have

\[ V(\tau_i, x) = x(\tau_i)^T P x(\tau_i) \]

\[ \leq \left( x(\tau_i) + \Delta x(\tau_i) \right)^T P \left( x(\tau_i) + \Delta x(\tau_i) \right) \]

\[ = x(\tau_i)^T P x(\tau_i) \leq 2x(\tau_i)^T P x(\tau_i) \]

where

\[ x(t-h_n(t)) = (x_1(t-h_n(t)), x_2(t-h_n(t)), \ldots, x_n(t-h_n(t)))^T. \]

Thus, condition (ii) of Lemma 1 is satisfied with \( g(s) = \eta s \).

For \( e < \infty \), we have the following two integrals

\[ G_i = \sup_{t \geq 0} \int_{t \geq 0} p(s) ds = \left( 1 + \mu^2 \right) \frac{1}{\lambda_n(P)} \tau, \]

And

\[ G_2 = \inf_{e > 0} \int_{e > 0} \frac{ds}{p(s)} = -\ln \eta. \]

By the inequality (6), the following inequality holds

\[ G_2 > G_i. \]

Thus, condition (iv) of Lemma 1 is also satisfied. Therefore, the system (5) is asymptotically stable.

We have \( \lim_{t \to \infty} x(t) = 0 \). By the inequality

\[ \|x(t) - 0\| \leq \|x(t)\|, \]

we know \( \lim_{t \to \infty} \|x(t) - x(t)\| = 0. \)

Then the state of system (2) tracks asymptotically and impulsively that of the reference system (1).
\[ +x(t)^{T} P B P^{-1} B^{T} P x(t) + \frac{1}{\eta} V(t) \leq \left( 1 + \lambda_{m}(E) + \frac{1}{\eta} + \frac{\mu^{2}}{\lambda_{m}(P)} \right) V(t). \]

Thus, condition (iii) is satisfied with \( p(t) = 1 + \lambda_{m}(E) + \frac{1}{\eta} + \frac{\mu^{2}}{\lambda_{m}(P)} \), and \( \gamma(s) = s \).

For \( \varepsilon < \infty \), we have the following two integrals

\[ G_{1} = \sup_{c=0}^{\infty} \int_{\gamma_{c}(s)}^{0} p(s)ds = \left( 1 + \lambda_{m}(E) + \frac{1}{\eta} + \frac{\mu^{2}}{\lambda_{m}(P)} \right) \tau, \]

And

\[ G_{2} = \inf_{\varepsilon > 0} \int_{\varepsilon}^{\infty} \frac{ds}{\varepsilon} = - \ln \eta. \]

By the inequality (6), the following inequality holds

\[ G_{2} > G_{1}. \]

Thus, condition (iv) of Lemma 1 is also satisfied. Therefore, the system controlled is asymptotically stable.

We have \( \lim_{t \to \infty} x(t) = 0 \). By the inequality

\[ \left\| x(t) - x(t) \right\| \leq l \left\| x(t) \right\|, \]

we know \( \lim_{t \to \infty} \left( x(t) - x(t) \right) = 0 \).

Then the state of system (8) tracks asymptotically and impulsively that of the reference system (1).

Remark 1. In view of Theorem 1 and its corollary, we see that under the control law \( \{ t_{e}, Q, x(t) - Q x(t) \} \), if the system (2) and (8) satisfy the conditions of Theorem 1, the state tracking performance index can realize.

Now, we investigate the possibility of designing impulsive control law based on measured output instead of the state information. Consider the state estimator of system (2) given by

\[ \ddot{x}(t) = A \dot{x}(t) + B \dot{x}(t - h_{e}(t)) + L C \left( x(t) - \dot{x}(t) \right) + w(x(t)), \]

\[ y(t) = C \dot{x}(t), \]

\[ \ddot{x}(t) = \ddot{x}(t), t \in [-h, 0]. \]

Where \( \ddot{x}(t) \) is the state estimator of \( x(t) \), \( L \) is the output feedback gain matrices, \( \ddot{x} \in C(R, R^{n}) \).

Define the difference between the real state and the estimator state as

\[ e(t) = x(t) - \ddot{x}(t). \]

From (2) and (9), we have

\[ e\dot{(t)} = (A - LC) e(t) + Be(t - h), \]

\[ y(t) = C e(t), \]

\[ e(t) = \phi(t) - \ddot{x}(t), t \in [-h, 0]. \]

For the asymptotic stability of system (10), we have the following theorem.

**Theorem 2.** The system (2) and system (9) is asymptotically synchronous if there exists output feedback matrix \( L \), such that the following equality holds,

\[ J = A + A^{T} + I - LC - C^{T} L^{T} + BB^{T} < 0. \]

**Proof.** Consider the following Lyapunov function candidate

\[ V(t, e(t)) = e(t)^{T} e(t) + \int_{t-h}^{t} e(s)^{T} e(s)ds \]

Obviously, \( V(t, e(t)) > 0. \)

By virtue of the Lyapunov function \( V(t, e(t)) \) upper Dini derivative along the solution of system (10), we have

\[ \dot{V}(t, e(t)) = e(t)^{T} e(t) + e(t)^{T} e(t) - e(t - h)^{T} e(t - h) \]

\[ = e(t) \left( A + A^{T} + I - LC - C^{T} L^{T} \right) e(t) \]

\[ + 2e(t)^{T} Be(t - h) - e(t - h)^{T} e(t - h) \]

\[ \leq e(t)^{T} \left( A + A^{T} + I - LC - C^{T} L^{T} + BB^{T} \right) e(t). \]

From the condition of theorem, we have

\[ V(t, e(t)) < 0. \]

Thus, the system (10) is asymptotically stable.

Now, we will solve problem is the state of estimator system (9) tracks asymptotically the one of reference system (1). If the difference between the real state and the estimator state is seen as the external disturbance of system (9), then, (9) can be rewritten as

\[ \ddot{x}(t) = A \ddot{x}(t) + B \ddot{x}(t - h) + \ddot{w}(t), \]

\[ \ddot{y}(t) = C \ddot{x}(t), \]

\[ \ddot{x}(t) = \ddot{x}(t), t \in [-h, 0]. \]

Where

\[ \ddot{w}(t) = LC \ddot{x}(t) + w(t). \]

For system (11), under the impulsive control law \( \{ t_{e}, Q, x(t) - Q x(t) \} \), we have the following result.

**Theorem 3.** Suppose that\n
\[ \left\| x(t) - \ddot{x}(t) \right\| \leq m \left\| \ddot{x}(t) \right\| \] and \( P > 0 \), if there exists output feedback matrices \( L \), such that the following two equality hold.

i) \( J = A + A^{T} + I - LC - C^{T} L^{T} + BB^{T} < 0 \),

ii) \( 0 < \eta \ddot{\eta} < e^{-\rho(t)} \).

Then the state of system (9) tracks asymptotically and impulsively that of the reference system (1).

Where

\[ \eta = 2 \left( \lambda_{m}(F) + \frac{\lambda_{u}(G)(1 + I)^{\frac{1}{2}}}{\lambda_{m}(P)} \right), F = I - 2Q + P^{-1}Q^{T} P Q, \]

\[ G = Q^{T} P Q, \]

\[ E = 2 \left( m^{2} \lambda_{m}(C^{T} L^{T} PLC + \mu^{2} \lambda_{m}(P)) \right)^{P^{-1}} \]

\[ + A + A^{T} + I - LC - C^{T} L^{T} + BB^{T} P. \]

**Proof.** Consider the following Lyapunov function candidate
\[ V(t, \bar{x}) = \mathbf{P}(t)^T \mathbf{P}(t) \]  

It clearly satisfies condition (i) of Lemma 1. 

At \( t = \tau_k \), \( k = 1, 2, \ldots \), we have 

\[ V(\tau_k, \bar{x}) \leq 2 \mathbf{P}(\tau_k)^T \left( P - 2PQ + Q^2 PQ \right) \mathbf{P}(\tau_k) + 2x_1(\tau_k)^T Q^2 PQ x_1(\tau_k)^T \leq 2 \left( \lambda_m(F) + \frac{\lambda_m(G) (1 + 1)}{\lambda_m(P)} \right) V(\tau_k, \bar{x}) \]

\[ \eta V(\tau_k, \bar{x}). \]

Thus, condition (ii) of Lemma 1 is satisfied with \( g(s) = \eta s \).

By virtue of the Lyapunov function (13) upper Dini derivative along the solution of system (2), it follows that at \( t = \tau_k \), \( k = 1, 2, \ldots \), we have 

\[ D^+ V(t, \bar{x}) \leq \mathbf{P}(t)^T \left( \mathbf{A}^T P + P \mathbf{A} + PB \mathbf{P}^{-1} B^T P \right) \mathbf{P}(t) + \mathbf{P}(t)^T \left( 2m^2 \lambda_m \left( C^T L P L C \right) I + 2\mu \lambda_m \left( \mathbf{P} \right) \mathbf{I} \right) \mathbf{P}(t) + \left( 1 + \frac{1}{\eta} \right) V(t) \leq \left( 1 + \frac{1}{\eta} + \lambda_m \left( E \right) \right) V(t). \]

Thus, condition (iii) is satisfied with \( \gamma(t) = s \) and \( p(t) = 1 + \frac{1}{\eta} + \lambda_m \left( E \right) \).

For \( \varepsilon < \infty \), we have the following two integrals 

\[ G_1 = \sup_{\varepsilon < \infty} \int_{t_0}^{t_\varepsilon} p(s) ds = \left( 1 + \frac{1}{\eta} + \lambda_m \left( E \right) \right) \varepsilon, \]

And 

\[ G_2 = \inf_{\varepsilon < \infty} \int_{t_0}^{\varepsilon} \frac{1}{G_1} p(s) ds = -\ln \eta. \]

By the inequality (12), the following inequality holds 

\[ G_1 > G_2. \]

Thus, condition (iv) of Lemma 1 is also satisfied.

Therefore, the system controlled (11) is asymptotically stable.

We have \( \lim_{t \to \infty} \overline{x}(t) = 0 \). By the inequality 

\[ \| x(t) - \overline{x}(t) \| \leq \| \mathbf{P}(t) \|, \]

we know \( \lim_{t \to \infty} \| x(t) - \overline{x}(t) \| = 0 \).

Then the state of system (9) tracks asymptotically and impulsively that of the reference system (1).

Remark 2. In view of Theorem 3, we see that when the system (10) is asymptotically stable, we can adopt the estimated system (9) to realize the state tracking performance index by impulsive control law \( \{ \tau_k, Q, x_1(t) - \overline{Q}(t) \} \).

Remark 3. Theorems 1 and 3 show that our designed impulsive controller for the realization of state tracking performance relates with the reference system and the tracking system.

IV. NUMERICAL EXAMPLE

In this section, we will consider two examples to illustrate the results obtained in Section 3. Consider the reference system (1) and the tracking system with the following data:

\[ A = \begin{bmatrix} -1.5 & -1.2 \\ -2 & -0.2 \end{bmatrix}, \quad A = \begin{bmatrix} 1.5 & -1 \\ -1 & -2.3 \end{bmatrix}, \]

\[ B = \begin{bmatrix} -0.3 & -0.2 \\ 0.1 & -0.4 \end{bmatrix}, \quad C = \text{diag}(2.3, -0.7), \]

\[ w(t) = -\left[ \frac{2}{3} x_1(t), \frac{4}{3} x_1(t), x_2(t) \right]^T, \]

\[ x_1(0) = (2, 1)^T, \quad x(0) = (-2, 1)^T, \]

\[ \overline{x}(0) = (-1.5, 1.4)^T, \phi(t) = (t + 2, t + 1)^T, \]

\[ \phi(t) = (t - 1.5, t + 0.6)^T. \]

Figure 1 and Figure 2 is the time sequence chart for system (1).

Figure 3 is the time sequence chart of the error system between system (1) and system (2), and Figure 4 is the time sequence chart of the error system between system (1) and state estimator system of system (2). They show that system (2) or state estimator system of system (2) can’t track asymptotically system (1) without impulsive control. Figure 5 is the time sequence chart of the error system between system (2) and its state estimator system.
Because this error system is asymptotically stable, our estimating for system (2) is very efficacious.

When the state of tracking system is measurable, obviously, \( \mu = 1 \). We can get the approximate value of \( l \) through simulations, we obtain \( l = 1 \).

We let

\[
Q_r = \text{diag} \left( \frac{1}{2(1+l)}, \frac{1}{2(1+l)} \right),
\]

\[
Q = \text{diag} \left( \frac{2}{3}, \frac{2}{3} \right), P = I, \tau = 0.01,
\]

Then, we have

\[
\eta = 0.7222, -\left( 1 + \frac{\lambda_M(E)}{\eta} \right) \tau = -0.069932.
\]

Obviously, \( \eta = 0.7222 < e^{-0.069932} = 0.9325 \).

Consequently, it follows from Theorem 1 that the state of system (2) tracks asymptotically the one of system (1) under the impulsive control law

\[
\left\{ r, Q, x, (t) - Qx(t) \right\}.
\]

When the state of tracking system is not measurable, obviously \( \mu = 1, \ m = 1 \), we can get the approximate value of \( l \) through simulations, we obtain \( l = 1 \).

We let

\[
Q_r = \text{diag} \left( \frac{1}{4}, \frac{1}{4} \right), Q = \text{diag} \left( \frac{2}{3}, \frac{2}{3} \right), P = I, \tau = 0.002, \tau = 0.002 \text{ and the output feedback gain matrices candidate } L = \text{diag} (1.63, -0.151).
\]

Then, we have

\[
\lambda_M \left( \lambda \right) = -1.5499 < 0, \text{ and}
\]

\[
-\left( 1 + \frac{\lambda_M(E)}{\eta} \right) \tau \approx -0.24086.
\]

Obviously, \( \eta = 0.7222 < e^{-0.2384} \approx 0.786 \).

Consequently, it follows from Theorem 1 that the state of system (2) tracks asymptotically the one of system (1) under the impulsive control law

\[
\left\{ r, Q, x, (t) - Qx(t) \right\}.
\]

The state tracking chart is shown in Figure 6. It is clear that the state of the state estimator system of system (2) tracks better the state of the system (1) after 0.015 second.

V. CONCLUSION

We proposed a design method for impulsive controller for the realization of state tracking performance in this paper. From the numerical example solved using this design method, we see that this impulsive controller is effective for the tracking system with time-delay and disturbance, which state is measurable or no-measurable.

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REFERENCES


