Modular Reversibility Analysis in Self-loops Connections of Petri Net Systems

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Abstract—It is well known that the use of a modular approach for modeling has many advantages: it allows the modeler to consider different parts of the model independently of one another. A modular approach to analysis is also attractive: it often dramatically decreases the complexity of the analysis task. To create Petri net models of large systems, four bottom-up techniques, consisting of sharing operation, synchronous operation, self-loops connection as well as inhibitor-arc connection, have been developed. This paper focus on the concurrent behavior relation in self-loops connections of Petri net systems. First, for the property of reversibility we show that it is possible to decide dynamic invariance of the global system from invariance of the individual modules. Second, for reversibility property we show that it is possible to construct reversibility of total modular system from reversibility of the individual modules without unfolding to the entire state space. Finally, we present some examples to illustrate the effectiveness of our approaches. The advantages of our approaches are in the context of concurrent language and can synthesize Petri net systems beyond asymmetric choice nets.

Index Terms—modular analysis, self-loops connection, reversibility preservation, Petri net systems

I. INTRODUCTION

Petri net system behavior depends on not only its graphical structure, but also on its initial marking, and it can be obtained through reachability analysis. The size of a reachability graph is determined by both the structure of system and the initial marking. In general, the larger structure of systems or the initial marking, the larger the reachability graph. And it has been shown that the complexity of the reachability analysis of Petri net system is exponential. Petri net systems composition can alleviate state space exploration by guaranteeing such good properties as liveness, deadlock-freeness, boundedness, reversibility and so forth while incrementally expanding the subsystems. Thus, composition operation is an effective way to manipulate systems of millions of states, and is playing an increasingly important role in theoretical and industrial fields. Normally, composition operations should obey the following three principles:

1. Preservation: The synthesized system should preserve some good properties.
2. Simplicity: The synthesis rules must be as simple as possible.
3. Generality: The rules should be as powerful as possible to generate as many classes of system as possible.

A lot of efforts has been done in this area. Wolfgang Reisig[1] provided the formal framework for a simple composition operator, adequate for many classes of Petri net applications. It requires a minimum of fairly intuitive technicalities from its users and readers. The modular state space technique[2] takes advantage of the modular organization of the model. Modular Petri nets consist only of modules synchronized through shared transitions. This modular approach can often decrease the complexity of the analysis task. A Petri net model is introduced in [3-4], which defines a set of basic subnets, namely elementary control tasks (ECT). Such a model can be applied to design logic controllers by bottom-up approach, and the subnets are used to model subsystems through a number of connection operations including self-loops, inhibitor-arc, and synchronization. The liveness preservation of Petri net in above operations of are discussed. The work of H.Q.Wang[5] studied system behaviors, namely, language of firing sequences, and investigated system behaviors in the synthesis of Petri net models by using operations of self-loops, inhibitors as well as synchronization. But their approaches are only based on sequential language not on concurrent language.

M.D.Jeng[6] proposed synthesis technique which allowed to model flexible manufacturing systems (FMS) guaranteeing property of liveness without posterior analysis. In his work, each subsystem is modeled as a resource control net module, and the net system is obtained by merging the modules conforming to two minimal restrictions and the system’s structural liveness is checked by an algorithm. The synthesis approach for modeling were described in [7-9]. These approaches can in fact guarantee the conservativeness of the synchronized net. The work of Aybar[10] proposed a decomposition technique with including principle which can be applied to decentralized control problem.

Z.J.Ding[14] introduced the refinement of Petri nets

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based on a k-well-behaved Petri net, in which k tokens can be processed. Then, according to the different compositions of subsystems, they also proposed well-, under- and overmatched refined Petri nets. In addition, the language and property relationships among sub-, original, and refined nets are studied to demonstrate behavior characteristics and property preservation in a system synthesis process.

L. Jiao[15] formulated resource sharing as a place fusion on a Petri net specification that satisfies a designated set of properties and includes some duplicated places representing accesses to the resources. Under some conditions, the obtained net will preserve the original properties after the incorporation of resource sharing. They considered two classes of property-preserving place fusions which can be applied to solve a resource sharing problem in the design of manufacturing systems.

Z. J. Ding[16] proposed an approach for modeling Web service composition by Petri nets which is based on OWL-S. By using a bottom-up approach, synchronous composition, the data flow net of the composite service is combined with the control flow net to obtain an integrated service net. Moreover, based on modeling approach, the boundedness and liveness properties of Petri net models are analyzed for guaranteeing the correctness of the composite Web service. Analysis and verification technique is based on the properties preservation criteria such that complexity is alleviated.

The remainder of this paper is organized as follows: Section 2 introduces some basic concepts and notations of Petri net system. Section 3 discusses the modular dynamic invariance in self-loops connection compositions. Section 4 gives some criteria which are necessary and sufficient for reversibility in self-loops connection processes. Section 5 discusses reversibility preservation for modular Petri net systems. Section 6 gives the remaining conclusions.

II. BASIC DEFINITION AND THEOREMS

A Petri net is a triple \( N = (P, T; F) \) such that \( P \) and \( T \) are disjoint finite sets, \( F \subseteq (P \times T) \cup (T \times P) \) and \( dom(F) \cup cod(F) = P \cup T \). The elements of \( P \) and \( T \) are respectively called places and transitions, and \( F \) is called the flow relation. The pair \((N, M_0)\) is a Petri net system, where \( M_0 \) is the initial marking.

**Definition 1:** Let \( \Sigma = (P, T; F, M_0) \) be a Petri net system, \( L_\Sigma(T) = \{ \alpha | \alpha \in T^* \} \) and \( M_0(\alpha >) \), \( L_\Sigma(\Sigma) \) is then called the sequential language of \( \Sigma \) or the sequential behavior of \( \Sigma \), where \( T^* \) is the closure of \( T \). Let \( L_\Sigma(C) = \{ \alpha | \alpha \in (T^*)^* \} \) and \( M_0(\alpha >) \), \( L_\Sigma(C) \) is called the concurrent language of \( \Sigma \) or the concurrent behavior of \( \Sigma \), where \( T^2 \) is the power set of \( T \). Usually, for \( R_t \in T^2 \), if \( |R_t| = 0 \), \( R_t \) is called an empty step and denoted by \( R_t = \lambda \). If \( |R_t| = 1 \), \( R_t \) is called a single step. If \( |R_t| > 1 \), \( R_t \) is called a concurrent step.

A step \( R_t \in T^2 \) is a set of transitions which can occur concurrently. For example, let \( R_t = \{ a, b \} \), it means that transitions \( a \) and \( b \) can occur simultaneously. We also represent \( R_t \) as \( \begin{pmatrix} a \\ b \end{pmatrix} \).

**Definition 2:** Let \( L \) be a concurrent language of \( \Sigma \), \( \alpha, \beta \) are step sequences of \( L \). We define two operations “\( \circ \)” and “\( + \)” as following:

\[
\alpha \circ \beta \equiv \alpha \beta \quad \text{and} \quad \alpha + \beta \equiv \{ \alpha, \beta \}
\]

Then the operation “\( \circ \)” and “\( + \)” are called connection and addition operations respectively.

Let \( \Sigma = (N, M_0) \) be a Petri net system, where \( N = (P, T; F) \). \( \Sigma \) is reversible iff (and only if) \( \forall M \in [M_0 >, M_0 \in [M >]. \) It means that for any marking \( M \) reached from the initial marking \( M_0 \), then the marking \( M_0 \) also can be reached from \( M \).

**Definition 3:**[12, 13] Let \( \Sigma_i = (P_i, T_i; F_i, M_0(i=1,2)) \) be Petri net systems, \( P_i \cap P_2 = \emptyset \) and \( T_1 \cap T_2 = \emptyset \). Let \( \Sigma = (P, T; F, M_0) \) be a Petri net system such that \( 1) P = P_1 \cup P_2, \quad 2) T = T_1 \cup T_2, \quad 3) F = F_1 \cup F_2 \cup \{(p^*, t_{i+1}^-, (t_{i+1}^-, p^*) | p^* \in P_i, t_{i+1}^-, t_{i+1} \in T_3\} \), \( i = 1,2 \). There is a self-loops between \( (p^*, t_{i+1}^-) \). \( 4) M_0(p) = M_0(p) \) if \( p \in P_i, i = 1,2 \). Then \( \Sigma \) is called a self-loops connection net system with \( \Sigma_1 \) and \( \Sigma_2 \), denoted by \( \Sigma_1 \cup \Sigma_2 \).

Let \( T_0 = \{ t^{(i)} | t^{(i)} \in T \} \) such that \( (p^{(3-i)}, t^i) \in F \land (t^i, p^{(3-i)}) \in F, i = 1,2 \) and \( T_0 = T_0 \cap T_1 (i = 1,2) \). If \( t \in T_0, t \) is called a synchronous transition of \( T \). Analogously, if \( t \in T_0, t \) is called a synchronous transition of \( T_1 \).

Let \( \Sigma = (P, T; F, M_0) \) be a Petri net system. For \( \forall P \subseteq P \), we represent \( M_1 \vert_{P'} \) as a restriction of \( M \) on \( P' \).

For \( \forall T \subseteq T, \alpha \in (T^*)^* \), we denote \( \alpha \vert_{P'} \) as a restriction of \( \alpha \) on \( P' \).

**Definition 4:** Let \( \Sigma_i = (P_i, T_i; F_i, M_0(i=1,2)) \) be Petri net systems, \( \Sigma = \Sigma_1 \cup \Sigma_2 = (P, T; F, M_0) \).

1) A step sequence \( \alpha \in (2T_i^*)^* \) is called a path on \( \Sigma_i \), if \( \exists M, M' \in [M_0 > \text{ such that } M(\alpha > M') \).

2) A step sequence \( \alpha \in (2T_i^*)^* \) is called a path on \( \Sigma \), if \( \exists M, M' \in [M_0 > \text{ such that } M(\alpha > M') \).

3) For an arbitrary path \( \alpha_1 \in (2T_i^*)^* \) on \( \Sigma \), with \( \alpha_1 \cap (2T_i^*)^* = \emptyset \), written as \( M_1(\alpha_1 > M_2, \text{ where } M_1 \in [M_0 > \text{ if there exists a path } \beta \in (2T_i^*)^* \) on \( \Sigma_3 \) satisfying \( \beta \cap (2T_i^*)^* = \emptyset \) such that:

\( 1) M_2(\beta > M_3 \) and \( 2) \exists R_t \in (2T_i^*)^*, M_3[R_t >. \)

then, \( \alpha = \alpha_1 \circ R_t \) is called a basic synchronous path (BSP) on \( \Sigma_i \), \( \beta \) is called a mutual synchronous path (MSP) of \( \alpha \) on \( \Sigma_3 \), and \( \alpha, \beta \) are called a pair of mutual synchronous paths on \( \Sigma \). Let \( R_t = (\alpha, \beta) \), then \( \alpha \) can be represented by \( \alpha = \alpha_1 \circ (\alpha, \beta) \) (note that \( \alpha \) and \( \beta \) might be \( \emptyset \)).

4) Let \( \alpha \) be a path on \( \Sigma \) satisfying \( \alpha \cap (2T_i^*)^* = \emptyset \), \( \alpha \) is called a basic nonsynchronous path (BNP) on \( \Sigma_i \).

**Definition 5:** Let \( \Sigma_i = (P_i, T_i; F_i, M_0(i=1,2)) \) be Petri net systems, \( \Sigma = \Sigma_1 \cup \Sigma_2 = (P, T; F, M_0) \), and let \( \alpha, \beta \) be two paths on \( \Sigma \), then we define an operation \( \otimes \) to represent the concurrent composition of paths on \( \Sigma \) shown as follows:
Remark 1: 1) $M \in [M_0 >]$; 2) The definition is a recursive one since case 1) is applied to the latter cases; 3) We apply operation $\cap$ to express concurrent composition of paths on $\Sigma$; 4) We provide two rules for calculus of paths on $\Sigma$ as follows:

4.1) The operation degree of $\cap$ is higher than $\cup$;

4.2) The operation degree of $\cap$ is higher than $\cup$.

Let $\Sigma_i = (P_i, T_i, F_i, M_0)$ $(i = 1, 2)$ be Petri net systems, $\Sigma = \Sigma_1 \cup \Sigma_2 = (P, T, F, M_0)$. (1) Let $B_{M,M'}^i = \{\alpha \in (2^{|T_i|})^*| \alpha$ is a BSP on $\Sigma_i$ satisfying $M[\alpha > M'],$ where $M \in [M_0 >]$ $(i = 1, 2)$. $B_{M,M'}^i$ represents the set of all BSFs on $\Sigma_i$ from $M$ to $M'$. (2) Let $B_{M,M'} = \{\alpha \in (2^{|T_i|})^*| \alpha$ is a BSP on $\Sigma_i$ satisfying $M[\alpha > M',$ where $M \in [M_0 >]$ $(i = 1, 2)$. $B_{M,M'}^i$ represents the set of all BSFs on $\Sigma_i$ from $M$ to $M'$. (3) Let $l_i(\Sigma_i) = \{B_{M,M'}^i|\forall M \in [M_0 >, \forall M' \in [M >], then $l_i(\Sigma_i)$ represents the set of all BSFs on $\Sigma_i$ $(i = 1, 2)$. (4) $l_i(\Sigma_i) = \{B_{M,M'}^i|\forall M \in [M_0 >, \forall M' \in [M >], then $l_i(\Sigma_i)$ represents the set of all BSFs on $\Sigma_i$ $(i = 1, 2)$.

Definition 6: Let $\Sigma_i = (P_i, T_i, F_i, M_0)$ $(i = 1, 2)$ be Petri net systems, $\Sigma = \Sigma_1 \cup \Sigma_2 = (P, T, F, M_0)$.

1) If each BSF on $\Sigma_i$ can be extended to a BSP on $\Sigma_i$, then $\Sigma_i$ is called a well Petri net system (WPNPS), i.e. for each BSF $\alpha$ on $\Sigma_i$, there exist $\alpha_1 \in (2^{(|T_i| \setminus \gamma_i)}^*)^* \cap R_t$ and $\beta \in (2^{(|T_i| \setminus \gamma_i)}^*)^*$ such that $\alpha \circ \alpha_1 \circ \beta$ is a BSP on $\Sigma_i$ and $\beta$ is its MSP, where $\setminus$ is a subtract operation of sets.

2) If $\Sigma_i$ is not WPNPS, $\Sigma_i$ is called an unwell Petri net system.

Definition 7: Let $\Sigma = (P, T, F, M_0)$ be Petri net system and $U, V$ the set of step sequences of $\Sigma$. If the language $L(\Sigma)$ of $\Sigma$ satisfies:

$L(\Sigma) = L(\Sigma) \setminus U + V$

then this equation is called a recursive language equation of $\Sigma$ or the language $L(\Sigma)$ can be iteratively generated by this equation, where the initial value is $L(\Sigma) = \{\lambda\}$.

Example 1: $\Sigma$ is synthesized with $\Sigma_1$ and $\Sigma_2$ by two self-loops connections, shown by figure 1 (From left to right, they are $\Sigma_1$, $\Sigma_2$ and $\Sigma$ respectively).
It is easily known that the set of all $BSP\alpha$ on $\Sigma_1$ is $l_1(\Sigma_1) = \{a, da, cda, bcd\}$ and the set of all $BNP\alpha$ on $\Sigma_1$ is $l_1(\Sigma_1) = \{e, b, c, d, bc, cd, bcd\}$. Similarly, it can be easy to verify that the set of all $BSP\alpha$ on $\Sigma_2$ is $l_2(\Sigma_2) = \{f, ef, hef, ghef\}$ and the set of all $BNP\alpha$ on $\Sigma_2$ is $l_2(\Sigma_2) = \{e, g, h, gh, he, ghe\}$.

From figure 1, we have $\alpha = a(b) f(\text{cd}) gh(abc)$ is a sentence of $L_c(\Sigma)$ and it can be shown that $\alpha = a \circ e \circ b \circ ef \circ cda \circ gh \circ bc$, where $a \in l_1(\Sigma_1), e \in l_2(\Sigma_2), b \in l_2(\Sigma_1), ef \in l_2(\Sigma_2), cda \in l_1(\Sigma_1), gh \in l_2(\Sigma_2), bc \in l_1(\Sigma_1)$. Because

$$\alpha = a \circ e \circ b \circ ef \circ cda \circ gh \circ bc$$
$$= a \circ b \circ ef \circ cda \circ gh \circ bc$$
$$= a \circ b \circ ef \circ cda \circ gh \circ bc$$

Furthermore, we have the following theorem:

**Theorem 1:** Let $\Sigma_i = (P_i, T_i; F_i, M_0_i)$ (i=1,2) be Petri net systems, $\Sigma = \Sigma_1 \cup \Sigma_2 = (P; T; F, M_0)$. Then the language $L_c(\Sigma)$ of $\Sigma$ (For sake of brevity, we write $L(\Sigma)$ as $L_c(\Sigma)$ in the rest of this paper) satisfying the following recursive language equation:

$$L(\Sigma) = L(\Sigma_1) \cup l_1(\Sigma_1) \cup l_2(\Sigma_1) \cup l_1(\Sigma_2) + l_2(\Sigma_2) \cup l_1(\Sigma_2) + l_1(\Sigma_1) + l_2(\Sigma_2))$$

where $l_1(\Sigma_1), l_2(\Sigma_1)$ (i=1,2) and $l_2(\Sigma_2)$ are defined by Definition 5. We denote this recursive language equation as the recursive equation (*).

### III. MODULAR DYNAMIC INVARIANCE OF PETRI NET SYSTEMS

The concept of dynamic invariance including state and behavior invariance was proposed in [13] in studying of synchronous and sharing synthesis processes. Their formal definitions are as follows:

**Definition 8:** Let $\Sigma_i = (P_i, T_i; F_i, M_0_i)$ (i=1,2) be Petri net systems, $\Sigma = \Sigma_1 \cup \Sigma_2 = (P; T; F, M_0)$, where $O$ is a synthesis operation. If $\forall M \in [M_0] >, M(\Sigma) \in [M_0_i] > (i=1,2)$, then the composite system $\Sigma$ satisfies state invariance.

**Definition 9:** Let $\Sigma_i = (P_i, T_i; F_i, M_0_i)$ (i=1,2) be Petri net systems, $\Sigma = \Sigma_1 \cup \Sigma_2 = (P; T; F, M_0)$, where $O$ is a synthesis operation. If $\forall M \in L(\Sigma), \alpha, \tau \in L(\Sigma)$ and $\alpha \circ \tau \in L(\Sigma)$, then the resultant system $\Sigma$ satisfies behavior invariance.

In paper[5], it showed that the dynamic invariance holds in a synchronous synthesis process except for sharing process. We now show that the synthesized system $\Sigma$ in self-loops connection with $\Sigma_1$ and $\Sigma_2$ also satisfies dynamic invariance.

**Theorem 2:** Let $\Sigma_i = (P_i, T_i; F_i, M_0_i)$ (i=1,2) be Petri net systems. $\Sigma = \Sigma_1 \cup \Sigma_2 = (P; T; F, M_0)$, then $\Sigma$ satisfies state invariance.

**Proof:** For $M \in [M_0] >$, written as $M_0(\alpha > M$.

We prove this theorem in following cases:

Case 1) if $\alpha$ is a $BNP\alpha$ on $PN_1$ (i=1,2), obviously $M_0(\alpha > M(\Sigma_1) and $M_0(\alpha > M(\Sigma_2)$. On the other hand, $M_0(\alpha > M_{p_{\Sigma_1}} = M_{p_{\Sigma_2}}$. Thus, $M(\Sigma) \in [M_0] > and $M(\Sigma_2) \in [M_0] >$.

Case 2) if $\alpha$ is composed of $BNP\alpha$ on $PN_1$ and $PN_2$, in turn, for sake of brevity, let $\alpha = \alpha_1 \circ \alpha_2 \neq \lambda$, where $\alpha_1$ and $\alpha_2$ are $BNP\alpha$ on $PN_1$ and $PN_2$, respectively. We similarly obtain $M_0_p(\alpha_1 > M(\Sigma_1) and $M_0(\alpha_2 > M(\Sigma_2)$. With 1), 2) we have $M(\Sigma) \in [M_0] > and $M(\Sigma_2) \in [M_0] >$.

Case 3) if $\alpha$ is composed of some pairs of $MSP\alpha$, let $\alpha = \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_2k-1 \circ \alpha_2k(\neq \lambda) (k \geq 1)$, where $\alpha_2j-1, \alpha_2j$ is a pair of $MSP\alpha$ $(1 \leq j \leq k)$. By induction over $k$, we discuss $\alpha$ in following cases:

Case 3.1) if $k = 1, \alpha = \alpha_1 \circ \alpha_2 \neq \lambda$. Suppose $\alpha_1$ is a $BSP\alpha$ on $PN_1$ and $\alpha_2$ is a $MSP\alpha$, written as $\alpha_1 = \alpha_1 \circ (\alpha_1 \circ \alpha_2)$, then we get $M_0(\alpha > M(\Sigma_1) and $M_0(\alpha > M(\Sigma_2)$. As $PN_1$ is an inhibitor-arc connection net system of $PN_1$ and $PN_2$, we have $M(\Sigma_1) = M_{P_{\Sigma_1}} = M(\Sigma_2)$ and $M(\Sigma_1) = M(\Sigma_2)$. Then

$$M(\Sigma_1) \in [M_0] > and M(\Sigma_2) \in [M_0] >$$

From 1), 2) we get $M(\Sigma) \in [M_0] > and M(\Sigma_2) \in [M_0] >$.

Case 3.2) if $k = n, the induction hypothesis is true.

Case 3.3) if $k = n + 1, let M_0(\alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_2n-1 \circ \alpha_2n > M(\Sigma_1) \circ \alpha_2n+2 > M, where \alpha_2n+1 \neq \lambda$. From the hypothesis, we have $M(\Sigma_1) \in [M_0] > and M(\Sigma_2) \in [M_0] >$. Suppose that $\alpha_2n+1$ is a $BNP\alpha$ on $PN_1$ and $\alpha_2n+2$ is its $MSP\alpha$, written as $\alpha_2n+1 = \alpha_2n+1 \circ (\alpha_2n+1 \circ \alpha_2n+2)$, then we obtain $M_0(\alpha_1 > M_0(\alpha_2n+1 > M, and $M_0(\alpha_2n+2 > M(\Sigma_1) or $M_0(\alpha_2n+2 > M(\Sigma_2)$. With 1), 2) we have $M(\Sigma) \in [M_0] > and M(\Sigma_2) \in [M_0] >$.

Note that $M(\Sigma_1) \in [M_0] > and M(\Sigma_2) \in [M_0] >$, then we get $M(\Sigma) \in [M_0] > and M(\Sigma_2) \in [M_0] >$.

Case 4) if $\alpha$ is composed of some pairs of $MSP\alpha$ and a $BNP\alpha$ on $PN_1$, written as $\alpha = \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_2k-1 \circ \alpha_2k \circ \alpha_2k+1(\neq \lambda), where \alpha_2j-1, \alpha_2j$ is a pair of $MSP\alpha$ $(1 \leq j \leq k)$ and $\alpha_2k+1$ is a $BNP\alpha$ on $PN_1$. Let $M_0(\alpha > M(\Sigma_1) \circ \alpha_2k \circ \alpha_2k+1(\neq \lambda, and $M_0(\alpha > M(\Sigma_2)$. As $\alpha_2k+1$ is a $BNP\alpha$ on $PN_1$, $M(\Sigma_1) \circ \alpha_2k \circ \alpha_2k+1 > M(\Sigma_2)$. From case 3.3), we have $M(\Sigma) \in [M_0] > and M(\Sigma_2) \in [M_0] >$.
Case 5) if $\alpha$ is composed of some pairs of $\text{MSP}_i$ and $\text{BNP}_i$ on $P N_{\alpha, \text{I}}$, $P N_{\alpha, \text{II}}$, with similarity to case 4), the conclusion can be easily proved.

**Theorem 3:** Let $\Sigma_i = (P_i, T_i; F_i, M_0_i)$ ($i=1,2$) be Petri net systems, $\Sigma = \Sigma_1 \otimes \Sigma_2 = (P, T; F; M_0)$, then $\Sigma$ satisfies behavior invariance.

**Proof:** This theorem in fact has been proved in the proof of theorem 2.

**IV. DETERMINATION OF REVERSIBILITY FOR TOTAL MODULAR PETRI NET SYSTEM**

To present a sufficient and necessary criterion for reversibility in self-loops operations, we need to introduce some new concepts:

**Definition 10:** Let $\Sigma = (P, T; F; M_0)$ be a Petri net system, $X$ be a $T$-invariant (i.e. $DX = 0$, where $D$ is incident matrix of $\Sigma$) and $X > 0$. Let $|X| = \{t_i | t_i \in T$ and $X(i) > 0\}$, then $|X|$ is called the support of $X$.

**Definition 11:** $X$ is called a minimal support of $T$-invariant iff there exists no $T$-invariant $X' (X' \neq X)$ such that $|X'| \subseteq |X|$. $X$ is called a minimal $T$-invariant iff $X$ is a minimal support of $T$-invariant and there exist no minimal support of $T$-invariant $X' (X' \neq X)$ such that $X' < X$.

$X' < X$ means that each component of $X'$ is less than its corresponding component of $X$, and there exist at least one component of $X'$ which is strict less than its corresponding component of $X$.

**Lemma 1:** Let $\Sigma = (P, T; F, M_0)$ be a Petri net system, if $\Sigma$ is reversible, then there exists $X > 0$ such that $DX = 0$.

**Proof:** The result is obvious.

**Definition 12:** Let $\Sigma = (P, T; F, M_0)$ be a Petri net system, $X$ is a minimal $T$-invariant of $\Sigma$. If there exists $\alpha \in (2^T)^*$ such that $M_0 \alpha > M_0$ and $\tau (t_i/\alpha) = X(i)$, where $\tau (t_i/\alpha)$ represents the occurrence number of $t_i$ in $\alpha$, $i = 1, \ldots, |T|$, then $\alpha$ is called a minimal reversible path with respect to $X$. If only $M_0 \alpha > M_0$ holds, then $\alpha$ is a called a reversible path.

**Definition 13:** Let $\Sigma_i = (P_i, T_i; F_i, M_0_i)$ ($i=1,2$) be Petri net systems, $\Sigma = \Sigma_1 \otimes \Sigma_2 = (P, T; F, M_0)$, $P(\Sigma_i)$ is the set of all minimal reversible paths in $\Sigma_i$, the set $P(\Sigma_i)$ can be in the following two cases:

$$P_C = \{\alpha | \alpha \in P(\Sigma_i) \text{ and } \alpha \cap 2^T \neq \emptyset\}$$

$$P_V = \{\alpha | \alpha \in P(\Sigma_i) \text{ and } \alpha \cap 2^T = \emptyset\}$$

**Definition 14:** Let $\Sigma = (P, T; F, M_0)$ be a Petri net system, if there is a vector $Y > 0$ such that $DY \neq 0$ and $|Y| \subseteq |X|$ for all minimal $T$-invariant $X$, then $Y$ is called an irreversible vector. If there exists $\alpha \in (2^T)^*$ such that $M_0 \alpha > 0$ and $\tau (t_i/\alpha) = Y(i)$, where $Y$ is an irreversible vector, $i = 1, \ldots, |T|$, then $\alpha$ is called an irreversible path with respect to $Y$.

**Lemma 2:** Let $\Sigma = (P, T; F, M_0)$ be a Petri net system.

1) $\Sigma$ is reversible iff there is no irreversible path in $\Sigma$.

2) $\Sigma$ is irreversible iff there is at least an irreversible path in $\Sigma$.

**Proof:** It can be proved by Definition 14 and Lemma 1 straightforwardly.

**Definition 15:** Let $\Sigma = (P, T; F, M_0)$ be a Petri net system, for any step sequence $\alpha \in (2^T)^*$, let $\text{pref}(\alpha) = \{\alpha | \alpha = \alpha \circ \alpha', \alpha \text{ might be } \varepsilon\}$, then $\text{pref}(\alpha)$ is called prefix set of $\alpha$, $\alpha$ is prefix sequence of $\alpha$.

**Theorem 4:** Let $\Sigma_i = (P_i, T_i; F_i, M_0_i)$ ($i=1,2$) be Petri net systems, $\Sigma = \Sigma_1 \otimes \Sigma_2 = (P, T; F, M_0)$, $\Sigma$ is reversible iff one of the following three cases holds:

**Case A:**

1) Both $\Sigma_i$ ($i=1,2$) are irreversible, and for any irreversible path $\alpha(i) \in L(\Sigma_i)$ ($i=1,2$), there exists no $\alpha(i-1) \in L(\Sigma_i)$ such that $\alpha(i) \circ \alpha(i-1) \neq \varepsilon$.

2) For $\forall \alpha(i) \in P(\Sigma_i), \forall \alpha(i-1) \in P(\Sigma_{i-1})$, ($i=1,2$), if $\alpha(i) \circ \alpha(i-1) = \varepsilon$, then for $\forall \beta(i) \in \text{pref}(\alpha(i))$, $\forall \beta(i-1) \in \text{pref}(\alpha(i-1))$, $\beta(i-1) \neq \alpha(i-1), \beta(i-1) \neq \varepsilon$, it holds that for $\forall \sigma(i-1) \in P(\Sigma_{i-1})$ with $\sigma(i-1) \neq \alpha(i-1), \beta(i-1) \notin \text{pref}(\sigma(i-1))$, then $\beta(i) \otimes \beta(i-1) = \varepsilon$.

3) there exists $\alpha(i) = a_1^1 \cdots a_{m+1}^{1} \cdots a_1^i \cdots a_{m+1}^{i} \cdots a_1^m$ such that $\alpha(i) \circ \alpha(i-1) \neq \varepsilon$, where $a_k^j \in P(\Sigma_i)$ ($j=1, \ldots, m$) and $a_k^i$ or $\alpha(i-1)$ might be $\varepsilon$ ($i=1,2$).

**Case B:**

1) $\Sigma_i$ is reversible, but $\Sigma_{i-1}$ is not ($i=1,2$), and for any irreversible path $\alpha(i) \in L(\Sigma_{i-1})$, there exists no $\alpha(i) \in L(\Sigma_i)$ such that $\alpha(i) \circ \alpha(i-1) \neq \varepsilon$.

2) For $\forall \alpha(i) \in P(\Sigma_i), \forall \alpha(i-1) \in P(\Sigma_{i-1})$, ($i=1,2$), if $\alpha(i) \circ \alpha(i-1) = \varepsilon$, then for $\forall \beta(i) \in \text{pref}(\alpha(i))$, $\forall \beta(i-1) \in \text{pref}(\alpha(i-1))$, $\beta(i-1) \neq \alpha(i-1), \beta(i-1) \neq \varepsilon$, it holds that for $\forall \sigma(i-1) \in P(\Sigma_{i-1})$ with $\sigma(i-1) \neq \alpha(i-1), \beta(i-1) \notin \text{pref}(\sigma(i-1))$, then $\beta(i) \otimes \beta(i-1) = \varepsilon$.

3) there exists $\alpha(i) = a_1^1 \cdots a_{m+1}^{1} \cdots a_1^i \cdots a_{m+1}^{i} \cdots a_1^m$ such that $\alpha(i) \circ \alpha(i-1) \neq \varepsilon$, where $a_k^j \in P(\Sigma_i)$ ($j=1, \ldots, m$) and $a_k^i$ or $\alpha(i-1)$ might be $\varepsilon$ ($i=1,2$).

**Case C:**

1) Both $\Sigma_i$ ($i=1,2$) are reversible.

2) For $\forall \alpha(i) \in P(\Sigma_i), \forall \alpha(i-1) \in P(\Sigma_{i-1})$, ($i=1,2$), if $\alpha(i) \circ \alpha(i-1) = \varepsilon$, then for $\forall \beta(i) \in \text{pref}(\alpha(i))$, $\forall \beta(i-1) \in \text{pref}(\alpha(i-1))$, $\beta(i-1) \neq \alpha(i-1), \beta(i-1) \neq \varepsilon$, it holds that for $\forall \sigma(i-1) \in P(\Sigma_{i-1})$ with $\sigma(i-1) \neq \alpha(i-1), \beta(i-1) \notin \text{pref}(\sigma(i-1))$, then $\beta(i) \otimes \beta(i-1) = \varepsilon$.

3) there exists $\alpha(i) = a_1^1 \cdots a_{m+1}^{1} \cdots a_1^i \cdots a_{m+1}^{i} \cdots a_1^m$ such that $\alpha(i) \circ \alpha(i-1) \neq \varepsilon$, where $a_k^j \in P(\Sigma_i)$ ($j=1, \ldots, m$) and $a_k^i$ or $\alpha(i-1)$ might be $\varepsilon$ ($i=1,2$).

**Proof:** $\Rightarrow$ Suppose $\Sigma$ is reversible, we discuss it in following three cases.

Case A: $\Sigma_i$ ($i=1,2$) are both irreversible. If for each irreversible path $\alpha(i)$, there exists $\alpha(i-1) \in L(\Sigma_{i-1})$ such that $\alpha(i) \circ \alpha(i-1) \neq \varepsilon$, then $\alpha(i) \circ \alpha(i-1)$ is obviously not irreversible path. From Lemma 2, $\Sigma$ is not reversible which contradicts with the assumption we made. Hence, condition 1 of Case A holds.

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Similarly, for $\forall a^{(i)} \in P(\Sigma_{\gamma})$ and $\forall a^{(3-i)} \in P(\Sigma_{\delta})$ (i=1 or 2), suppose $a^{(i)} \otimes a^{(3-i)} = \varepsilon$, then for $\forall a^{(i)} \in \text{pref}(a^{(i)})$, $\forall a^{(3-i)} \in \text{pref}(a^{(3-i)})$ ($a^{(i)} \neq a^{(3-i)}$, $\beta^{(i)} \neq \varepsilon$), if for $\forall a^{(i)} \in P(\Sigma_{\gamma})$ ($a^{(i)} \neq a^{(3-i)}$), $\beta^{(i)} \neq \text{pref}(a^{(3-i)})$ and $\beta^{(i)} \otimes \beta^{(3-i)} \neq \varepsilon$ hold, then $\beta^{(i)} \otimes \beta^{(3-i)}$ is not a reversible path in $\Sigma$.

By Lemma 2, $\Sigma$ is not reversible which also contradicts with the assumption we made, thus condition 2 of Case A holds.

Since $\Sigma$ is reversible, there must be a reversible path $\alpha$ in $\Sigma$. Let $\alpha^{(i)} = \alpha_{T_{i}}$ (i=1,2), then due to $M_{0}P_{i} = M_{0}$ we have $M_{0}\alpha^{(i)} > M_{0}$, and $M_{0}\alpha^{(3-i)} > M_{0}$ hold in single $\Sigma_{i}$ and $\Sigma_{3-i}$ respectively from Theorem 3. If $\alpha^{(i)} \neq \varepsilon$, then $\alpha^{(i)}$ is a reversible path. The same as $\alpha^{(3-i)}$. So, condition 3 of Case A holds.

Case B and Case C can be proved in similar ways.

We only prove Case C holds, other cases can be demonstrated analogously. Suppose $\alpha$ is an irreversible path in $\Sigma$. Let $\alpha^{(i)} = \alpha_{T_{i}}$ (i=1,2), then $\alpha = \alpha^{(i)} \otimes \alpha^{(3-i)} \neq \varepsilon$. If $\alpha^{(i)} \neq \varepsilon$, then $\alpha^{(i)}$ is a reversible path in $\Sigma_{i}$ because there is no irreversible path in $\Sigma_{i}$ from the reversibility of $\Sigma_{i}$. Analogously, if $\alpha^{(3-i)} \neq \varepsilon$, then $\alpha^{(3-i)}$ is a reversible path in $\Sigma_{3-i}$. Let $\alpha^{(i)} = \sigma_{i}^{(1)} \sigma_{i}^{(2)} \ldots \sigma_{i}^{(m)}$, where $\sigma_{i}^{(j)} \in P(\Sigma_{i})$ (i=1, \ldots, m), $\sigma_{i}^{(j)} \subseteq \beta^{(i)}$ and $\beta^{(i)} \in P(\Sigma_{i})$. Let $\alpha^{(3-i)} = \sigma_{3-i}^{(1)} \sigma_{3-i}^{(2)} \ldots \sigma_{3-i}^{(m)}$, where $\sigma_{3-i}^{(j)} \in P(\Sigma_{3-i})$ (k=1, \ldots, m), $\sigma_{3-i}^{(j)} \subseteq \beta^{(3-i)}$ and $\beta^{(3-i)} \in P(\Sigma_{3-i})$. If $\sigma^{(i)} = \varepsilon$ or $\sigma^{(i)} = \beta^{(i)}$, $\sigma^{(3-i)} = \varepsilon$ or $\sigma^{(3-i)} = \beta^{(3-i)}$, then $\alpha = \alpha^{(i)} \otimes \alpha^{(3-i)} \neq \varepsilon$ is a reversible path in $\Sigma$ which contradicts with the irreversibility of $\Sigma$. We now suppose $\sigma^{(i)} \neq \varepsilon$ and $\sigma^{(3-i)} \neq \varepsilon$.

(1) Assume $\sigma^{(i)} \subseteq \beta^{(i)}$ and $\sigma^{(3-i)} \subseteq \beta^{(3-i)}$, let $\alpha' = (\sigma_{i}^{(1)} \ldots \sigma_{i}^{(m)} \beta^{(i)}) \otimes (\sigma_{3-i}^{(1)} \ldots \sigma_{3-i}^{(m)} \beta^{(3-i)})$. If $\alpha' \neq \varepsilon$, then $\alpha'$ is a reversible path and $\alpha' \subseteq \alpha$. On the other hand, $\alpha$ is an irreversible path in $\Sigma$. Then $\alpha \not\subseteq \alpha'$. We get a contradiction. Hence, $\alpha' = \varepsilon$. It is easy to known that either $\beta^{(i)} \in P_{T_{i}}(\Sigma_{i})$, $\beta^{(3-i)} \in P_{T_{i}}(\Sigma_{3-i})$ or $\beta^{(i)} \in P_{T_{i}}(\Sigma_{i})$, $\beta^{(3-i)} \in P_{T_{i}}(\Sigma_{3-i})$ satisfy $\beta^{(i)} \otimes \beta^{(3-i)} = \varepsilon$. With condition 2, $\Sigma_{i} \otimes \Sigma_{3-i} = \varepsilon$. Thus, $\alpha = (\sigma_{i}^{(1)} \ldots \sigma_{i}^{(m)}) \otimes (\sigma_{3-i}^{(1)} \ldots \sigma_{3-i}^{(m)}) (\neq \varepsilon)$ is a reversible path in $\Sigma$. We have a contradiction.

(2) Similar as proved in (1), we obtain that if $\sigma^{(i)} \subseteq \beta^{(i)}$ and $\sigma^{(3-i)} \subseteq \beta^{(3-i)}$, $\alpha$ is a reversible path in $\Sigma$. We get a contradiction.

(3) Similar as proved in (1), we obtain that if $\sigma^{(i)} \subseteq \beta^{(i)}$ and $\sigma^{(3-i)} \subseteq \beta^{(3-i)}$, $\alpha$ is a reversible path in $\Sigma$. We have a contradiction.

Overall, there is no irreversible path in $\Sigma$. From condition 3, there is at least one reversible path in $\Sigma$. Hence, $\Sigma$ is reversible by Lemma 2.

We now present four examples to illustrate Theorem 4.

Example 2: $\Sigma$ is a self-loops connection net with $\Sigma_{1}$ and $\Sigma_{2}$ shown by Figure 2.

It is easy to know that $\Sigma_{1}$ is reversible; $t_{1}t_{2}t_{3}t_{4}$ is a minimal reversible path in $\Sigma_{1}$; $t_{6}t_{8}t_{5}$ and $t_{4}t_{6}t_{5}$ are minimal reversible paths in $\Sigma_{2}$. It can be verified that $t_{1}\otimes t_{2}\otimes t_{3}\otimes t_{4}(t_{6}\otimes t_{8}) = \varepsilon$, and there exists $\beta^{(1)} = t_{1}t_{2}t_{3} \in \text{pref}(t_{1}t_{2}t_{3}t_{4})$, $\beta^{(2)} = t_{6} \in \text{pref}(t_{6}t_{8})$ such that for $\forall \sigma \in P(\Sigma_{2}) (\sigma \neq t_{6}t_{8})$, we have $\sigma = t_{1}t_{2}t_{3}$, $t_{6} \notin \text{pref}(\sigma)$ and $(t_{1}t_{2}t_{3}) \otimes t_{6} \neq \varepsilon$. Therefore, condition 2) of case A, B, C does not hold. On the other hand, $(t_{1}t_{2}t_{3}) \otimes t_{6}$ is an irreversible path in $\Sigma$, that is, $t_{1}t_{2}t_{3}$ is an irreversible path in $\Sigma$.

Example 3: $\Sigma$ is a self-loops connection net with $\Sigma_{1}$ and $\Sigma_{2}$ shown by Figure 3.

Example 4: $\Sigma$ is a self-loops connection net with $\Sigma_{1}$ and $\Sigma_{2}$ shown by Figure 4.
conditions of Case C are all satisfied, $\Sigma$ is reversible by Theorem 4.

Example 5: $\Sigma$ is a self-loops connection net with $\Sigma_1$ and $\Sigma_2$ shown by Figure 5.

![Figure 5. Illustration of Case C of Theorem 4](image)

It easy to know that both $\Sigma_1$ and $\Sigma_2$ are reversible. $t_2t_5, t_1t_3$ are minimal reversible paths in $\Sigma_1$, $t_7t_5t_9t_9, t_7t_8t_9t_9$ and $t_7 \left( \frac{t_6}{t_8} \right)$ $t_9$ are reversible paths in $\Sigma_2$. It can be verified that all conditions of Case C are satisfied, then $\Sigma$ is reversible from Theorem 4.

V. REVERSIBILITY PRESERVATION IN MODULAR PETRI NET SYSTEMS

The reversibility preservation guarantees that the reversibility of subsystems make the reversibility of the global system.

Definition 6: Let $\Sigma = (P,T;F,M_0)$ be a Petri net system, if for $t \in T$, there exists $t' \in T(\neq t)$ such that $\alpha(t') = t, \alpha(t) = \alpha(t')$, then $t'$ is called a substituted transition of $t$.

Theorem 5: Let $\Sigma_i = (P_i,T_i;F_i,M_{0i})$ be Petri net systems, $\Sigma = \Sigma_1 \cup \Sigma_2 = (P,T;F,M_0)$. If the following conditions hold, then $\Sigma$ is reversible.

1) $\Sigma_i (i = 1,2)$ are reversible.
2) for $\forall \alpha \in P_i(\Sigma_i)$ ($i = 1,2$), then all synchronous transitions in $\alpha$ have their substituted transitions and there is at least one substituted transition is not synchronous.

Proof: According to Case C of Theorem 4, we need to prove conditions 2 and 3 of Case C hold. From the given condition 1), we have $P(\Sigma_i) \neq \emptyset$ ($i=1,2$) from the given condition 1). Furthermore, for $\forall \alpha^{(i)} \in P(\Sigma_i)$ ($i=1,2$) we have $M_{0i}(\alpha^{(i)}) > M_{0i}$ holds in single $\Sigma_i$. After replacing each synchronous transition of $\alpha^{(i)}$ with a substituted transition (changed to be $\alpha^{(i)}$), we also obtain $M_{0i}(\alpha^{(i)}) > M_{0i}$ holds in single $\Sigma_i$. By the same way, for $\forall \alpha^{(3-i)} \in P(\Sigma^{(3-i)})$, its corresponding $\alpha^{(3-i)}$ also satisfies that $M_{0(3-i)}(\alpha^{(3-i)}) > M_{0(3-i)}$ and $\alpha^{(3-i)} \in P(\Sigma^{(3-i)})$. Let $\alpha = \alpha^{(i)} \cup \alpha^{(3-i)}$, then $\alpha$ is a reversible path in $\Sigma$. From the given condition 2), for $\forall \alpha^{(i)} \in P(\Sigma_i), \forall \alpha^{(3-i)} \in P(\Sigma)\cup P(\Sigma^{(3-i)})$, we have $\alpha^{(i)} \cup \alpha^{(3-i)} \neq \emptyset$. Hence, the Case C of Theorem 4 always holds. So, $\Sigma$ is reversible.

Example 6: $\Sigma$ is a self-loops connection net with $\Sigma_1$ and $\Sigma_2$ shown by Figure 6.

![Figure 6. Illustration of Theorem 5](image)

It easy to know that $\Sigma_1$ and $\Sigma_2$ are reversible. $t_2t_5t_9 \in P(\Sigma_1)$ is the minimal reversible path in $\Sigma_1$ and synchronous transition $t_4$ has a substituted transition $t_9$. On the other hand, $t_9$ is not a synchronous transition. Similarly, $t_8t_9t_10 \in P(\Sigma_2)$ is the minimal reversible path in $\Sigma_2$ and synchronous transition $t_9$ has a substituted transition $t_6$. On the other hand, $t_9$ is not a synchronous transition. From Theorem 5, $\Sigma$ is reversible.

Theorem 6: Let $\Sigma_i = (P_i,T_i;F_i,M_{0i})$ ($i = 1,2$) be Petri net systems, $\Sigma = \Sigma_1 \cup \Sigma_2 = (P,T;F,M_0)$. If the following conditions hold, then $\Sigma$ is reversible.

1) $\Sigma_i (i = 1,2)$ are reversible.
2) all synchronous transitions of $\Sigma_i$ ($i = 1,2$) are in the minimal reversible paths of $\Sigma_i$.
3) there is at most one self-loop arc between an arbitrary minimal reversible path in $\Sigma_i$ and an arbitrary minimal reversible path in $\Sigma_{3-i}$. Two minimal reversible paths connected with each self-loop arc are not the same paths, and moreover, the synchronous transitions connected with each self-loop arc are not the same transitions.

Proof: According to Case C of Theorem 4, we need to prove conditions 2 and 3 of Case C hold. From the given condition 1), we have $P(\Sigma_i) \neq \emptyset$ ($i=1,2$). If both $P(\Sigma_i) \neq \emptyset$ and $P(\Sigma_{3-i}) \neq \emptyset$, then there exist $\alpha^{(i)} \in P(\Sigma_i)$, $\alpha^{(3-i)} \in P(\Sigma_{3-i})$ such that $\alpha^{(i)} \cup \alpha^{(3-i)} \neq \emptyset$, i.e. $\alpha$ is the minimal reversible path in $\Sigma$. Without loss of generality, suppose $P(\Sigma_i) = \emptyset$ and $P(\Sigma_{3-i}) = \emptyset$. Then, for $\forall \alpha^{(i)} \in P(\Sigma_i)$ ($i = 1,2$), from the given condition 2) we have $\alpha^{(3-i)} \in P(\Sigma_{3-i})$ and there is a self-loop arc between $\alpha^{(i)}$ and $\alpha^{(3-i)}$. From the given condition 3) we obtain $\alpha = \alpha^{(i)} \cup \alpha^{(3-i)}$ can always occur in $\Sigma$. Thus, condition 3 of Case C holds. From the given conditions 2) and 3), it is easy to prove that condition 2 of Case C also holds. Then, the conclusion is obtained.

VI. CONCLUSION

Reversibility is a significant property of systems which represents the ability of systems returning to the initial marking from any marking reached from an initial marking. Usually, reversibility analysis is performed using reachability graph reduced by stubborn sets. Due to the explosion problem of the state space of reachability graph, the production of the minimal reachability graph by stubborn sets is also a NP-hard problem. Modular methods provide a solution to alleviate the large state space that needed to be explored when analyzing such properties as reversibility, liveness, and deadlock-freeness etc, of large systems. This paper focus on modular reversibility analysis in self-loops connection operations.
the methods are based on concurrent language not on sequential language. We present an operation "$\otimes$" to express the concurrent composition of paths and establish a recursive language equation to judge the reversibility of synthesized system. We also demonstrate an important property, namely, dynamic invariance, in self-loops operations. Moreover, we propose a criterion, which is sufficient and necessary for reversibility of global system and present some conditions to preserve reversibility in self-loops connection processes. One of the main advantages of our approaches is that we can synthesize Petri net systems beyond asymmetric choice net systems. In particular, the approaches presented here can easily be generalized to Petri net systems with weighted arcs.

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