Partial Shape Matching of 3D Models Based on the Laplace-Beltrami Operator Eigenfunction

Lijun Jiang*, Xutang Zhang, and Guangyu Zhang
School of Mechatronics Engineering, Harbin Institute of Technology, Harbin 150001, China
Email: 2836478@qq.com, zxt@hit.edu.cn, zgyqyx@hope.hit.edu.cn
*Corresponding author

Abstract—The comparison and matching based on partial description of the 3D model is the current focus of study in the shape analysis. The partial description based on the eigenfunction of the Laplace-Beltrami operator is an important way. A large number of eigenfunction values of any point on the surface of the model form a eigenvector; based on this vector, K-means clustering method will be used to query the model which is divided into several regions; for each region, based on the Hungarian method which is used in the solving of optimal assignment problem, search a corresponding region in the compared model, so that achieving the partial matching between the two models.

Index Terms—Partial Matching; Laplace-Beltrami Operator; Region Segmentation

I. INTRODUCTION

Under the circumstance of the rapid spread of application of the 3D model modeling software and more and more 3D models which can be shared on the Internet, how to reuse and manage the existing 3D mesh model and how to find the existing 3D mesh model according to the new design goal, that is the local matching and retrieval of 3D model, has become an important research topic [1][2].

Partial matching of the 3D model is related to the segmentation of the 3D model, and thoughts based on the description and retrieval of the segmentation of 3D model is from the basic principles of cognitive psychology. Psychology [3] and psychophysics [4] indicate that people’s recognition of shape is mostly based on the segmentation. Complex model is usually seen as the combination of simple basic elements or components, and the significant local shape features shield other non-significant characteristics with higher advantage. In order to get significant local features, 3D model must be split. The segmentation of the 3D model shape means that in the describing and matching problems of the 3D model shape, for example, before the retrieval, firstly the segmentation is carried out in accordance with certain geometric characteristic of the 3D model (such as contour line or geodesic, etc.) or specific algorithm, and then respectively calculate the shape characteristics of the model subblocks after the segmentation, and construct the descriptors of local features.

The study on the segmentation of 3D model is not yet mature currently, and most of the work exists some defects, for example, excessive segmentation, meaningful segmentation results can not be got, the segmentation results are not easy for parameterization, the algorithm is limited to face the particular application and specific shape model, and so on [5].

The segmentation technology of 3D model has been widely used in various research areas, such as surface deformation [6], mesh parameterization [7], collision detection [8], skeleton extraction in animation [9] and so on. The relative typical algorithm in the segmentation algorithm is the watershed segmentation algorithm, which was first applied to the image processing [10]. Reference [11] and other people expand the watershed segmentation technology in image processing to the 3D space. Reference [12] puts forward a fast marching watershed algorithm in accordance with the minimum rule, which is used to improve the segmentation result in 3D space.

The study on the segmentation and matching of 3D model of this article is carried out based on the above context. As to the 3D models, first apply LB operator to obtain the feature descriptor of model vertices, then adopt the K mean clustering method to cluster the vertex of a model and divide the region; as to each regions, search a corresponding region in another model based on the Hungarian method which is used in the solving of optimal assignment problem, to achieve the partial and global matching between two models.

The paper is organized as follows: some related results in the field of 3D model matching are reviewed in the next section. In Section 3 we describe the application of the Laplace-Beltrami operator on 3D model. Additionally, we describe a new feature descriptor of 3D model based on LB operator eigenfunctions.

In Section 4 we use K-Mean algorithm to decompose the 3D model, and descript every region with descriptor prompt by Section 3.

In Section 5 we present the Hungary method to calculate the similarity of the two models.

Section 6 describes experiments with several data sets, and Section 7 draws some conclusions and proposes directions for future work.

II. RELATED WORK

Methods such as Laplacian Eigenmaps [13], commonly used for dimensionality reduction, are better
to find a mapping that best preserves relationship between the 3D models.

A common approach to partial shape matching consists in finding local area similarity of the shapes, and performing the matching with 3D models. One methods, Zhang and Jain [14] proposed matching the embeddings of the 3D models into a spectral domain, using a nonrigid variant of the Iterative Closest Point algorithm. Another method which proposed by Kimmel and Elad [15], embeds the 3D models into a flat Euclidian space, and the Euclidian distances between vertices in the plane space approximate the geodesic distances on the shape surface.

W. Chang and M. Zwicker [16] and Huang and B. Adams et al. [17] proposed a method that modeling the non-rigid transformation between the 3D models by a group of rigid transformations applied to parts of the 3D models.

Mateus and Horaud [18], and Knossow and Sharma et al. [19], used spectral embedding to performing registration in the embedding domain with a probabilistic framework.

Hu and Hua [20] and Zaharescu et al. [21] proposed methods to detect similarity between nontrivial feature points on the 3D models. They employ the Laplace-Beltrami decomposition to find features.

The method proposed by Rustamov [22] using the eigendecomposition of the Laplace-Beltrami operator to construct an isometric invariant surface representation, and it’s aiming for correspondence detection, not to measure similarity between 3D models.

Method by Anguelov and Srinivasan et al. [23] defined a probabilistic model by the set of all possible correspondences, and match the shapes with belief propagation technique.

In a recent year, Lipman and Funkhouser [24], treated isometric transformations as a subset of Mobius transformations, and detected correspondences in the canonical domain of Mobius invariant representations.

The GMDS (Generalized Multidimensional Scaling) algorithm [25] also yields correspondence between two given 3D models.

Keriven and Thorstensen [26] extended the GMDS method to surfaces with textures. But the methods require significant computation efforts.

Other methods for non-rigid model shape comparison [25, 27, 28] used the Gromov-Hausdorff distance [29] to measure the similarity between the the 3D model shapes.

Other matching methods include the algorithm proposed by Sheffer and Zhang et al. [30], based on combinatorial tree traversal for similarity search;

Tevs and Bokeloh et al. [31] proposed a new method based on a geodesic distance for preserving randomized feature matching.

Another, more general, Leordeanu and Hebert in [32] proposed a spectral technique for correspondence problems using pairwise constraints. It is based on a spectral technique, and employs local descriptor similarity and global pairwise correspondence assignment quality to measure the matching cost.

Ovsjanikov and Sun et al. [33] showed that intrinsic symmetry detection can reduce the external symmetry detection in the domain of GPS (Global Point Signature) embedding of the 3D model surface.

Raviv et al. [34] used GMDS [25] to solve the symmetry detection as a problem of embedding a 3D model shape into itself.

Our main contributions can be summarized as follows:

We propose a descriptor of 3D model with the eigenfunctions of the Laplace-Beltrami operator and restrict the first K eigenfunctions for the descriptor.

Then we use K-Mean clustering algorithm to divide the 3D model into several regions with the descriptor based on Laplace-Beltrami operator.

Finally for each region, based on the Hungarian method which is used in the solving of optimal assignment problem, search a corresponding region in the compared model, so that achieving the partial matching between the two models.

III. SHAPE DESCRIPTION BASED ON LAPLACE-BELTRAMI OPERATOR EIGENFUNCTION

In differential geometry, the Laplace operator can be generalized to the definition of the function operator on the curved surface or more generally Riemannian manifold with pseudo-Riemannian manifold. This more general operator is called Laplace - Beltrami operator. Laplace - Beltrami operator is defined as the divergence of gradient as same as Laplace operator.

Laplacian Eigen maps, a good non-linear data dimensionality reduction method, was put forward by Mikhail Belkin and Partha Niyogi, it considers that the neighboring point in the high-dimensional space should also be contiguous when it is mapped to low-dimensional space [35]. The core part of the algorithm is very simple, involving only local computations and a sparse eigenvalue problem. The neighboring figure which is constructed in the algorithm reflects the intrinsic geometric structure of manifold, at the same time, it can find optimal linear approximate solution of the eigen equation of Laplace-Beltrami operator on the manifold.

The eigenvalues and eigenvectors of the Laplace-Beltrami operator which is defined on the differentiable manifold M satisfy the following equation:

$$\Delta f = -\lambda f$$  \hspace{1cm} (1)$$

According to the self-adjoint and semi-positive definiteness of the Laplace-Beltrami operator, the solution of the above eigen equation only has one; the orthogonal eigen system $$B = \{(\lambda_i, \psi_i), (\lambda_j, \psi_j)\}$$ is a pair of eigenvalues and eigenvectors. Among which

$$\Delta \psi_i = \lambda \psi_i, \quad \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_i \leq \lambda_{i+1} \leq \ldots.$$  

The corresponding eigenvectors of different feature values are orthogonal to each other, namely:

$$\langle \psi_i, \psi_j \rangle = \int_M \psi_i \psi_j = 0$$  \hspace{1cm} (2)$$

The characteristic system constructs the base of secondary integrable function space which is defined on the manifold M.
The general form of the Laplace-Beltrami operator with the discretization of smooth function \( f \) which is defined on the model \( M \):

\[
\Delta f(p_i) = \frac{1}{d_i} \sum_{j \in N(i)} w_{ij} [f(p_i) - f(p_j)]
\]

(3)

In the formula, \( N(i) \) is one ring of neighboring regions of the vertex \( p_i \), \( d_i \) is the quality of vertex, \( w_{ij} \) is symmetrical edge weights. The discretization matrix representation of the characteristic equation of above Laplace-Beltrami operator is the following generalization feature value problem:

\[
Lf = \lambda Df
\]

(4)

In the formula, \( f = (f(p_1), f(p_2), ..., f(p_n))^T \), which means the value of \( f \) at each vertex; the stiffness matrix \( L = V - W \); the weighted matrix \( W = (w_{ij}) \); the diagonal matrix \( V = diag(v_1, v_2, ..., v_n) \), \( v_i = \sum_{j \in N(i)} w_{ij} \); the mass matrix \( D = diag(d_1, d_2, ..., d_n) \).

Same as the Laplacian operator, the eigenfunction \( f \) is isometric constant for the curved surface \( M \), so the eigenfunction \( f \) can be used as the descriptor of 3D model:

\[
f^M(v) = \{f_1^M(v), f_2^M(v), f_3^M(v), ..., \} \in M
\]

(5)

Among which, \( f_k^M(v) \) is the value of the point \( v \) of the \( k \)-th eigen equation of Laplacian operator on the model \( M \).

Because the computational complexity of the matching depends on the element count of the descriptors. So we restrict the dimension of the descriptors to the first \( K \) eigenfunctions instead of using all the eigenfunctions of the Laplace-Beltrami operator.

In our experiments we performed a quality matching with a small number of eigenfunctions (\( K = 5 \) to 10).

Figure 1 shows segmentations induced by the descriptor composed by the 2\(^{nd} \)-7\(^{th} \) eigenfunctions, with our method.

Figure 1. Segmentations induced by our method with the 2\(^{nd} \)-7\(^{th} \) eigenfunctions.

According to the selection strategy of edge weights \( w_{ij} \) and vertex quality \( d_i \), the Laplace-Beltrami operator with the discretization can be divided into the geometric discrete operator and finite element discrete operator [36]. According to the literature [37], finite element discrete operator has a better approximation of accuracy and robustness as compared with the geometric discrete operator, so the finite element operator can be used for the discretization of the Laplace-Beltrami operator. Here we use the cotangent weights as the weighting matrix \( W \) and the vertex associated area as the mass matrix \( D \), which is expressed as follows:

\[
W(i,j) = \frac{\cot \alpha_i + \cot \beta_j}{2} \quad (i, j) \text{ edge}
\]

(6)

\[
D(i,j) = \frac{|t_i| + |t_j|}{12} \quad (i, j) \text{ edge}
\]

(7)

\[
\alpha_i, \beta_j, t_i, t_j \text{ is shown in Figure 1, and area means the area of the triangle. Among which, } |t_i| \text{ is the area of triangle } t_i, t_j \text{ and } t_k \text{ are the area of two triangles including the edges which are constructed by point } i \text{ and point } j. \text{ } W \text{ is the stiffness matrix which is constructed based on the cotangent weights, and } D \text{ is the mass matrix.}
\]

IV. THE REGION SEGMENTATION BASED ON THE K-MEANS CLUSTERING

Cluster analysis is one of multivariate statistical analysis methods, and the goal of cluster analysis is to collect data on a similar basis to classify. Internationally, the 3D model clustering segmentation work includes the hierarchical clustering analysis method based on the grid model [38], K-mean clustering method [39], K-mean fuzzy clustering method [40] and so on. Prior to cluster analysis, the similarity between the data must be first analyzed. Cluster analysis on the 3D model usually measures the degree of similarity with the distance between the data, and each data includes a lot of attributes, forming an eigenvector, and the distance between the data is transformed into calculation of the distance between the two vectors. Commonly distance measures include Minkowski distance, Euclidean distance, absolute distance, Chebyshev distance, Mahalanobis distance, Lanberra distance and so on. Euclidean distance is used most commonly among a variety of ranging algorithm, so we use the Euclidean distance as the ranging function of cluster analysis.
Before the clustering, we must firstly define the distance between classes, and the differences of distance definition between classes lead to different system clustering methods. Common distance definition between classes have eight kinds, and the corresponding system clustering methods also have eight kinds, respectively is: the shortest distance method, the longest distance method, the middle distance method, the center of gravity method, the group average method, the variable category average method, the variable method and the square sum of deviations method. Their classification steps are basically same, and the main difference is the different method of calculating the distance between the classes. Thereinafter \( d_{ij} \) means the distance between sample \( \mathbf{X}_i \) and \( \mathbf{X}_j \), \( D_{ij} \) means distance between the class \( G_i \) and \( G_j \).

In 1969, Wishart proposed a general formula for the distance between the classes [41]. The class spacing between class \( C_p \) and \( C_q \) is supposed as \( D_{pq} \), and the class \( C_r \) is merged by subclass \( C_p \) and \( C_q \) to from \( C_r = C_p \cup C_q \). The general formula between class distances can be expressed as:

\[
D_{3} = \alpha_{rk} D_{pk} + \alpha_{qk} D_{qk} + \beta_{pq} + \gamma |D_{pk} - D_{qk}|
\]  

(8)

In the above formula, when the coefficient \( \alpha_{rk} \), \( \alpha_{qk} \), \( \beta \), \( \gamma \) are valued as 1/2, 1/2, 0, -1/2, the class distance formula of the shortest distance method is got; when the value is 1/2, 1/2, 0, 1/2, the class distance formula of the longest distance method is got; when the value is 1/2, 1/2, -1/4, 0, the class distance formula of the intermediate distance method is got; by setting up the value of four parameters, the class distance formula of centroid distance method, average distance method and square sum of deviations method can be respectively got, which is not listed in detail here.

In this article, the shortest distance method is used for 3D model clustering segmentation. The shortest distance method means that the distance between the two classes is defined as the smallest distance between all elements of a class and all elements of another class. After the merger of the two classes, the distance between it and other classes is the smallest in all distances, so making the space quickly shrink. We conduct the K-means clustering segmentation of 3D mesh model. Figure 2 is the clustering segmentation results on the model, and we specify \( K \) as 4, which has good segmentation results.

The final number \( K \) of clustering is taken in advance. In our experiment, when we specify the value of \( K \) is greater than or equal to 8, the loop terminates.

The K mean clustering process of 3D model is as follows:

First of all, select \( K \) eigenvectors \( f^1_i, f^2_i, \ldots, f^K_i \) as initial clustering vectors from the collection of eigenvectors to be classified. In accordance with the principle of minimum distance, differentiate each eigenvector into certain class of \( K \) class, that means if the distance between the eigenvector \( x_i \) and the center vector \( f^j_i \) of class \( G^j_i \), \( d^j_i = \min_{j} \left| d^j_i \right| \), then in the new clustering results, \( x_i \in G^{j+1}_i \). Recompute the new clustering center vector, and the new center vector is the mean vector of all eigenvectors in the class \( K \). If the new class heart is consistent with the last clustering class heart, or the number of clustering is greater than or equal to 8, then the clustering terminates.

Because K-mean clustering method [42] belongs to the dynamic clustering, under the premise of determined number of classes and selected initial clustering vectors, it is the optimum clustering of the mininum Euclidean distance between each eigenvector to the center of class it belongs. K-mean algorithm is a greedy algorithm, therefore it is not quite certain to get the optimal result, the result is partially optimal only. The experiments show that the K-mean clustering is simple and fast, and the clustering segmentation results of most 3D models are relatively satisfactory.

V. SIMILAR REGIONAL SEARCH

Hungary method is a kind of method to solve the minimal (optimization direction is minimum) assignment problem, which was proposed initially by W. W. Kuhn and formed after improvement, and the solution is named upon a theorem from Hungarian mathematician D. Konig. Its basic principle is: for any efficiency matrix \( (C_{ij}) \) of the minimum, add to or subtract from the same constant \( k \) to each element of certain row or column to obtain a new matrix \( (h_{ij}) \), and then the new matrix and the original matrix have the same optimal solution. If the problem is to seek the maximum value, then the objective function needs to be converted into the minimum, and then use the Hungarian method to get the solution.

The definition \( M_i \) and \( M_j \) are two models to be matched, wherein the definition \( X \) is one clustering segmentation result on the model \( M_x \), and \( Y \) belongs to one perfect matching of \( M_x \), \( x \in X, y \in Y \), and when \( x \) and \( y \) satisfy the following formula

\[
y' = \arg \min_{y \in Y} \left\| f^X(x) - f^Y(y) \right\|
\]  

(9)

The value of \( y' \) is minimum, we say that point \( x \) and \( y \) match. That is, \( x \) and \( y \) are the nearest neighboring points in \( X \) and \( Y \), and the absolute value of the eigenfunction value of model \( M_i \) and model \( M_j \) on the point \( x \) and point \( y \) is minimum. The definition set \( C \) is the matching point set \( \{x_n, y_m\} \).
If all vertices in \( X \) have a matching point in \( Y \), we say that \( X \) and \( Y \) match. Taking the symmetry of the model and the inevitable errors in the calculation into account, point \( x \) on \( X \) may exist a number of matching points on \( Y \), and in order to filter out certain symmetry and error matching points, select the matching point \((x_m, y_m)\) and \((x_n, y_n)\) in the collection \( C \) of paired points, then calculate the distance of two vertices \( x_m \) and \( x_n \) in \( X \) and the distance of two vertices \( y_m \) and \( y_n \) in \( Y \), and if the two distances are equal, these two vertices are considered to be matched. This method can be expressed with the following formula:

\[
A_{mn} = |d_x(x_m, x_n) - d_y(y_m, y_n)|
\]

(10)

In the above formula, \( A_{mn} \) represents the matching degree of the points on \( X \) against \( x_m \) and \( x_n \) and the points on \( Y \) against \( y_m \) and \( y_n \) and when \( A_{mn} \) equals to 0 or is close to 0, we can consider that the point against \( x_m \) and \( x_n \) and the point against \( y_m \) and \( y_n \) match. \( d(x_m, x_n) \) can be Euclidean distance or geodesic distance, here we choose the geodesic distance to calculate the distance between point \( x_m \) and \( x_n \) and the distance between point \( y_m \) and \( y_n \) on the model, which can better reflect the surface information of the model than the Euclidean distance.

As shown in the formula, \( A \) is a similarity matrix, and the Hungary method can be used for the calculation on \( A \) to obtain the direct matching relation between \( X \) and \( Y \) of \( A \). The calculation steps are as follows:

1. Transform the initial matrix \( A \): subtract the smallest element of each row of the matrix \( A \), and then subtract the smallest element of each column in the matrix, making each row and each column exist 0 element;
2. Suppose \( F \) as the matrix which is obtained in Step 1, and then change 0 element in matrix \( F \) into 1 element, then non 0 elements is changed into 0 element, matrix \( G \) is obtained;
3. Select the row containing the least 1 element in all rows of \( G \), compare the number of 1 elements in the corresponding column, choose the 1 element in the column with the least 1 element, then clear “0” of the row and column with the selected 1 element; repeat step 2 and 3 until there is no position of 1 element, a independent 1 element group is obtained;
4. If the independent 1 element group is still not the largest independent 1 element group of the original matrix, then look for the extendable road method for the expansion, and the largest independent 1 element group of the original matrix can be got. If the number of 1 element is equal to the order of the matrix, then the largest match is got, calculation stops, otherwise going to the next step.
5. Increase more 0 elements for \( F \): make the least straight line to cover all elements of the matrix \( F \), then find the smallest element in the portion which is not covered by the straight line; subtract the smallest element from all rows which is covered by the straight line, and add the smallest element to each column which is covered by the straight line, a new matrix is obtained, return to step 2;
6. Mark the matrix in the maximum matching as \( Z \). When the element \( Z_{ij} \) in the matrix \( Z \) equals to 1, then the \( i \)-th point in the point set \( X \) and the \( j \)-th point in the point set \( Y \) are considered to be matching.

VI. EXPERIMENTAL RESULTS

The partial matching experiment is carried out for 2 triangular mesh models with the proposed method. The grid models in the experiment are taken from Purdue University ESB (Engineering SHAPE benchmark) database [43]. MATLAB software can achieve the solution of discrete Laplace-Beltrami operator as well as the matching and searching of K-means clustering and Hungary methods. When clustering, a number of eigenfunctions of Laplace-Beltrami operator can be selected, and select according to the difference between the feature values, that is, if the two feature values are very close, then discard one eigenfunction. The number of clustering is set to 3-7.

Follow is our two experiments with the method for segments and matching.

Figure 4. The segmentation result of the first model

Figure 5. The similar regional search result of the second model

Figure 6. The match result
Next Figure 7 and 8 and 9 shows our second experiments with T-Model.

![Figure 7. The match result](image)

Figure 7 shows the first T-Model was split into 5 segmentations, and the left one is the original model, the right one is the result.

![Figure 8. The match result](image)

Figure 8 shows the search result for the second T-Model and the left one is the original model and the right one is the result.

In Figure 9, the line shows the matching relations between two models.

![Figure 9. The match result](image)

Using the above method, we conduct several models retrieval on the ESB database, with the results shown in the figure 10.

![Figure 10. The match result](image)

VII. CONCLUSION

We proposed one method of local matching for two models, based on the eigenfunction of Laplace-Beltrami operator as the shape descriptor, through the solution of the generalization feature value problem, to obtain the vector description of the eigenfunction value of the point on the grid; with K-means clustering method, one model is first split, and then search for the corresponding part of each split region in another model. It not only achieves the matching of the local area between the models, but also gets the corresponding relation between the points in similar areas.

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REFERENCES


Lijun Jiang received his bachelor degree in School of Computer Science and Technology from Harbin Institute of Technology, Harbin, China in 2002 and received his master degree in School of Software from Harbin Institute of Technology, Harbin, China in 2004. Now he is studying in the School of Mechatronics Engineering, Harbin Institute of Technology, Harbin for his doctor degree. His research interests include 3D model retrieval and 3D model reconstruction.