Abstract—In order to improve the correlation accuracy of correlation algorithm, the fractional order correlation algorithm of uncertain time sequences is proposed in this paper. By taking advantage of the memory property of fractional order, the algorithm introduces the measurement of fractional order differential for the local trend of time sequence into the correlation algorithm and also analyzes the influences of differential order and noise upon correlation accuracy, provides selection relations between noise level and order. It has been proven with examples that the correlation accuracy of fractional order correlation algorithm has increased by one orders of magnitude as compared with slope correlation.

Index Terms—fractional order, uncertainty, time series, association

I. INTRODUCTION

Time series are a set of chronological observations of a certain indicator. The fuzzy membership of the description information and the import of the error message caused series of the description uncertainty. The correlation analysis of the uncertain time series can help us to overcome the limitation of the cognitive ability and ultimately find out the internal relations of different objectives.

The application of correlation analysis of uncertain time sequences is mainly embodied in factor analysis, decision making and superiority analysis. Correlation analysis is widely applied in numerous fields such as economy, society, agriculture, mining, transportation, education, medical science, ecology, environment, water conservancy, hydrology, petroleum, geology and aviation, solving lots of practical problems unsolved in the past. Since small sample and little information are the objective state of correlation analysis in practical application, this paper will focus on the study of correlation model of uncertain time sequences with multiple small samples and little information.

II. REVIEW

A. Correlation Algorithm Review

Grey correlation analysis highlights development tendency while pays less attention to sample size and typical regularity of distribution. Therefore, Ref.[1,2] grey correlation analysis has a great advantage for the correlation model of uncertain time sequences with multiple small samples and little information.

In recent years, Ref.[3-5] gray correlation algorithm obtained a significant development, and many scholars have made great contributions. From the relational degree itself, it experienced from the gray relational algorithm of no differential measurement information (such as Tang's correlation, the absolute correlation II, the relative correlation, correlation interval I, range correlation II) to gray relational algorithm with first-order differential metrical information (such as absolute relational degree I, slope correlation, and T-type correlation), and then turned to be the gray relational algorithm with second-order differential metrical information (B-type correlation, C-type correlation).

The abovementioned indicates that the introduction of the high order information and the fractional information into the associated metrics of the uncertain time series is the development trend of related algorithms.

B. Summary of Fractional Differential

Fractional calculus refers to the calculus with order of any real number order. For more than three centuries, many famous scientists did a lot of basic work on fractional calculus; however, fractional calculus really began to grow till the last 30 years. Ref.[6] Oldham and Spanier discussed mathematical calculations of the fractional number and their application in areas like physics, engineering, finance, biology, etc. Ref.[7] In 1993, Samko made systematic and comprehensive exposition on fractional integral and derivative related properties and their applications. Ref.[8] Many researchers have found that, fractional derivative model can more accurately describe the nature of a memory and the genetic material and the distribution process than integer order derivative model. The overall and memory characteristic of fractional are widely used in physics, chemistry, Ref.[9] materials, fractal theory, Ref.[10] image
processing and Ref.[11,12] other fields. Currently, the analysis of fractional differential has become a new active researched area that aroused great attention of domestic and foreign scholars, and turned to be the world’s leading edge and hot research field.

III. ALGORITHM PRINCIPLE

A. The Import of Fractional Order

Differential operations can enhance the high-frequency and weaken low-frequency of the signals. Fractional differential operation can nonlinearly improve more high-frequency and weaken less low-frequency of the signals with the growth of the order. From the perspective of the information extraction, the order of integer order operations is discrete whereas fractional one is continuous, and can provide more sequence information to help the identification of the sequence.

Each observed value on the time series is the common result of variety of subjective and objective factors and the development of all previous observations; therefore time series is of overall and memory characteristics. Fractional differential operator is intended differential operator with overall and memory characteristics whereas integral order doesn’t have this feature. Therefore, from the description of the time series it can conclude that, fractional differential could more accurately describe the memory nature of time series comparing with the integral order one, and was imported to calculate the relevancy of time series.

B. The Nature of Fractional Differential

Fractional differential operator can meet the exchange rate and the overlay standard $D^0 D^\nu s(t) = D^\nu D^0 s(t) = D^{\nu + 0} s(t)$, $(0, 1)$ differential order measures the overall situation of the sequence, other differential order results can all be acquired through the iterate integer-order differential on it. First-order differential reflects the slope of the sequence, second-order differential reflects the curvature of the sequence, and they all response to the partial trends of the sequence. To give consideration to the measurements of both global and local trends, non-integral order emphasis on $(0, 1)$ order, integer order taking into account of the first, second order, therefor, this paper is only analysis the $(0, 3)$-order differential related information.

C. Fractional Differential Difference Form

Since time series are discrete, when using the fractional differentials in it’s associate calculation, the definition pattern of fractional differential algorithms must be change into the difference form. Then, we derive the fractional differential difference formula via Grünwald-Letnikov definition.

Known, $v$ order fractional differential Grünwald-Letnikov definition is

$$G^a \frac{D^v}{dt} s(t) = \lim_{h \to 0} s^v_i(t)$$

$$= \lim_{h \to 0} \frac{h}{v} \sum_{r=0}^{n} C_{r}^{-v} s(t - rh)$$

$$C_{r} = \frac{\Gamma(-v+1)\ldots(-v+r-1)}{r!}$$

Where in:

According to Expression(1), if the persistent period of $s(t)$ is: $t \in [a, b]$, divide $[a, b]$ into equal parts corresponding to one unit interval $h=1$, it can be got

$$n = \left\lceil \frac{t-a}{h} \right\rceil = [t-a]$$

Then, $v$ order fractional differential difference expression to unitary signal $s(t)$ can be get:

$$\frac{d^v}{dt^v} s(t) \approx s(t) + (-v)s(t-1) + \frac{(-v)(-v+1)}{2} s(t-2)$$

$$+ \frac{(-v)(-v+1)(-v+2)}{6} s(t-3) + \ldots$$

$$+ \frac{\Gamma(-v+1)}{n!\Gamma(-v+n+1)} s(t-n)$$

From this differential expression, the difference coefficient of the fractional is:

$$a_0 = 1, a_1 = -v, a_2 = \frac{(-v)(-v+1)}{2}$$

$$a_3 = \frac{(-v)(-v+1)(-v+2)}{6}, \ldots, (2)$$

$$a_n = \frac{\Gamma(-v+1)}{n!\Gamma(-v+n+1)}$$

IV. Slope Correlation Algorithm Experiment

A. Slope Correlation Degree

$$r(X_0, X_1) = \frac{1}{n-1} \sum_{i=1}^{n} r_j(t), \quad 1 + \frac{\Delta x_0(t)}{x_0}$$

$$r_j(t) = \frac{\Delta x_0(t)}{x_0} + \frac{\Delta x_i(t)}{x_i} - \frac{\Delta x_j(t)}{x_j}$$

$$\bar{x}_0 = \frac{1}{n} \sum_{i=0}^{n} x_0(t)$$

$$\Delta x_0(t) = x_0(t+1) - x_0(t), \quad \bar{x}_j = \frac{1}{n} \sum_{i=1}^{n} x_j(t)$$

$$\Delta x_j(t) = x_j(t+1) - x_j(t)$$

B. Experimental Analyses

Assume $X_1, X_2, \ldots, X_7$ are time sequences, where
Assume \( X_i \) is the reference sequence, see Figure 1.

Add noise with range of \([-1, 1]\) into the reference sequences \( X_1 \) to get \( X_{11} \); add noise with range of \([-10, 10]\) into all the tracking to get \( X_{12} \), in which \( i=1, 2, ..., 7 \); add noise with range of \([-100, 100]\) into all the tracking to get \( X_{13} \), in which \( i=1, 2, ..., 7 \). Calculate respectively the slope correlation between \( X_{12} \) and \( X_{11} \), \( X_{13} \) and \( X_{11} \) (see figure 2 and 3). In the figure, \( r_{hk} \) stands for the correlation degree between \( x_h \) and \( x_k \), in which \( r \) stands for correlation degree, \( h \) and \( k \) are the subscripts of the sequences generated after adding noise.

IV. FRACTIONAL ORDER CORRELATION ALGORITHM

A. Fractional Order Correlation

Assume \( X_0 \) is the reference sequence, the degree of fractional order correlation between \( X_i, i=1, 2, ..., n \) and \( X_0 \) under \( v \) order is

\[
r(X_0, X_i, v) = \frac{1}{n-5} \sum_{k=6}^{n} r(x_0(k), x_i(k), v),
\]

\( v \in (0, 3) \)

\[
r(x_0(k), x_i(k), v) = (\min_i \min_k |x^v_0(k) - x^v_i(k)|)
+ 0.5 * \max_k \max_i |x^v_0(k) - x^v_i(k)|)
+ 0.5 * \max_i \max_k |x^v_i(k) - x^v_i(k)|
\]

\[
x^v_i(k) = \sum_{d=1}^{5} x_i(d) * a_{i-j-1}; i=0,1,...,n.
\]

Then calculate separately the curve of correlation degree (see figure 4,5) at the order of \((0, 3)\) between \( X_{12} \) and \( X_{11} \), \( X_{13} \) and \( X_{11} \).
From figure 4 it can be seen that the correlation value of rx1112 is far greater than the values of other curves when $v \in (0, 3)$. This indicates that when noise signal amplitude ratio is 0.01, fractional order correlation is accurate enough.

![Figure 5](image5.png)

**Figure 5.** Fractional Order Correlation Curve of Sequences $X_{13}$ and $X_{11}$

It can be seen from figure 5 that the correlation value of rx1113 is greater than the values of other curves when $v \in (0,1.5)$ and that the correlation value of rx1113 does not reach the maximum when $v \in (1.5, 3)$. This is because the increase of noise level has affected the correlation accuracy of the higher-order differential. When the order is set at 0.5, then the following fractional order correlation will be obtained (see figure 6).

![Figure 6](image6.png)

**Figure 6.** Fractional Order Correlation of Sequences $X_{13}$ and $X_{11}$ when the Order is 0.5

It can be seen from figure 6 that when noise signal amplitude ratio is 0.1 and the sequence order is 0.5, fractional order correlation is of high degree of accuracy.

### B. Correlation Judgment

1. Judgment for correlation values
   The greater the correlation value is, the greater the correlation between sequences is. Otherwise, the smaller the correlation between sequences is.

2. Relationship between order and correlation
   Comparing with high order differential, low order differential extract more of the low-frequency information and less high-frequency information. As for the time series, low order differential extract more long-term-effect information while high order differential extract more short-term-effect information.

   In the circumstance of no noise, the correlation accuracy will increase as the order increases. On the other hand, noise level will influence the correlation accuracy of the algorithm. The addition of noise will affect the high-frequency information of the sequence and as noise level increases, its influence upon correlation accuracy will expand from high order to low order.

   The selection relations between noise and order have been achieved through series of experiments. It is discovered that when noise signal amplitude ratio is 0.01, differential order is set at $(0.5, 2)$; when noise signal amplitude ratio is 0.1, differential order is set at 0.5. When noise signal amplitude ratio is unknown, the order is set at 0.5 and if the correlation values approximate each other, then the order needs to be increased until the correlation values become distinctive between one another. Therefore, the correlation accuracy of fractional order algorithm is suitable at best for the distinction of sequence correlation when noise signal amplitude ratio is 0.1.

   It can be concluded from the results of figure 2, 3, 4, 5 and 6 that the correlation accuracy of fractional order correlation algorithm had increased by one order of magnitude as compared with slope correlation algorithm.

### V. Conclusions

This paper proposes the fractional order correlation algorithm of uncertain time sequences, and analyzes the influences of differential order and noise upon correlation accuracy, provides selection relations between noise level and order. The experiments proved that fractional order correlation algorithm has increased by one orders of magnitude as compared with slope correlation.

### ACKNOWLEDGMENT

This work was financially supported by the National Natural Science Foundation of China (Grant No. 10731050) and Ministry of Education Innovation Team of China (No. IRT00742).

### REFERENCES


Ming-Liang Hou received his M.S. degree in China University of Petroleum in 2004 and Ph. D. degree in Institute of Optics and Electronics, Chinese Academy of Sciences in 2008. He is now a Lecturer in School of Computer Engineering, Huaihai Institute of Technology, Lianyungang, China. His current research interests ranged over the fields of Grey Theory, Pattern Recognition, Virtual Simulation, Intelligent Fault Diagnosis Technology.

Yu-Ran Liu received her M.S. degree in China University of Petroleum in 2004 and PhD degree in Institute of Optics and Electronics, Chinese Academy of Sciences in 2009. She is now an Associate Professor in School of Computer Engineering, Huaihai Institute of Technology, Lianyungang, China. Her current research interests lied in the fields of Grey Theory, Operational Research, Image Processing, Pattern Recognition, etc.

Yun Hu received her M.S. degree in Southeast University in 2007 and she is working towards her PhD. in the Department of Computer Science and Technology, Nanjing University. She is now a Lecturer in School of Computer Engineering, Huaihai Institute of Technology, Lianyungang, China. Her current research interests covered Uncertainty Theory, Data mining, Distributed artificial intelligence, etc.