Some Scoring Functions of Intuitionistic Fuzzy Sets with Parameters and Their Application to Multiple Attribute Decision Making

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Abstract—A novel intuitionistic fuzzy sets with parameters (IFSP) is introduced in this paper. Compared with conventional intuitionistic fuzzy sets (IFS), IFSP can provide more choices when it is applied to multiple attribute decision making. By analyzing the degree of hesitancy, an intuitionistic fuzzy sets with double parameters (IFSDP) model is presented. And then this paper proposes some weighted scoring functions based on IFS and IFSP. Finally, a multiple attribute decision making example applied to city planning is given to demonstrate the application of IFSDP and its weighted scoring functions. The simulation results show that the weighted scoring function method based on IFSDP is more comprehensive and flexible than the traditional intuitionistic fuzzy sets method.

Index Terms—intuitionistic fuzzy sets, intuitionistic fuzzy sets with parameters, scoring function, multiple attribute decision making

I. INTRODUCTION

In 1986, Atanassov introduced membership function, non-membership function, and hesitancy function, and presented IFS ([1]). Hence, many scholars applied IFS to decision-making analysis and pattern recognition widely. In the research field of IFS, Yager discussed the cut set characteristics of IFS in [3], Szmidt and Kacprzyk applied it to medical diagnosis in [4, 5, 6], some scholars (Z. S. Xu, H. Zhao, G. W. Wei, S. Lu, and X. Y. Yue et al.) applied it to decision-making analysis in [11, 12, 13, 14, 15, 20, 21, 22], some researchers (S. P. Xu, D. F. Li, Y. H. Li and W. L. Hung et al.) applied it to pattern recognition in [7, 8, 9, 10], Lei studied intuitionistic fuzzy reasoning in [17], and Y. Q. Zhang and X. B. Yang applied it to attribute reductions based on rough sets in [19].

By introducing membership function $\mu_{d}(x)$, non-membership function $\nu_{d}(x)$ and hesitancy function $\pi_{d}(x)$, the IFS theory is established, which generalizes Zadeh’s fuzzy sets (FS) ([1, 2, 3]). According to the IFS definition, $\mu_{d}(x)$ denotes the proportion of the support party, $\nu_{d}(x)$ denotes the proportion of the opposition party, and $\pi_{d}(x)$ denotes the proportion of the absent party. Though many scholars studied IFS and applied it to decision making ([11, 13, 15, 18, 20, 21, 22]), their methods are suitable for static model and unsuitable for dynamic model. Then Xu and Yager ([12]) presented a dynamic decision making model, which was also studied by Wei, Su, et al. ([14, 16]). However, traditional decision making researches based on IFS do not involve the detachment of the absent party, which means that the absent party is not specifically analyzed in conventional decision making model of IFS. Moreover, the absent party may change over time in practice, while conventional IFS method cannot deal with this kind of dynamic decision making model. Thus, the research on the variation of the absent party will play an important role in dynamic decision making. Taking into account this, we present a novel IFSP model by analyzing the hesitancy function.

We first assume that $\mu_{d}(x)$ is the firm support party of event A, $\nu_{d}(x)$ is the firm opposition party of event A, $\pi_{d}(x)$ is the maximum absent party of event A, $\pi^{*}_{d}(x)=(1-\lambda_{d}(x))\pi_{d}(x)$ is the firm absent party of event A, and $\pi_{d}(x)-\pi^{*}_{d}(x)=\lambda_{d}(x)\pi_{d}(x)$ denotes the convertible absent part, where $\lambda_{d}(x)$ is the proportion of the convertible absent individuals in all the absent individuals. Obviously

\[ \pi_{d}(x) = \begin{cases} \lambda_{d}(x) & \text{if } x \in A \setminus A_{0} \\ 0 & \text{if } x \in A_{0} \end{cases} \]
we have \( \mu_{A}(x) + \nu_{A}(x) + \pi_{A}(x) = 1 \) and \( 0 \leq \lambda_{A}(x) \leq 1 \). We divide the convertible absent part into two parts: \( \lambda_{A}(x)\nu_{A}(x)\pi_{A}(x) \) being the absent party which can be converted into the support party, and \( \nu_{A}(x)(1-\lambda_{A}(x))\pi_{A}(x) \) being the absent party which can be converted into the opposition party, where \( \lambda_{A}(x) \) is the proportion of the convertible absent individuals being converted to the support party, and \( 1-\lambda_{A}(x) \) is the proportion of the convertible absent individuals being converted to the opposition party. And we also have \( 0 \leq \lambda_{A}(x) \leq 1 \).

According to the detachment of the absent party, we propose a series of definitions and construction methods of IFSP, and concentrate on the model of IFSDP. Then, we introduce a type of scoring function on the basis of IFS and generalize it to IFSP. And then taking advantage of the scoring function, IFSDP is applied to multiple attribute decision making. We can adjust the parameters to appropriate values to obtain all the feasible results.

Therefore, the IFSP method can be applied to the dynamic decision making field, and we can predict all the possible decision making results in the future according to the variation of membership function, non-membership function, and hesitancy function. In summary, the new method raised in this paper can expand the scope of IFS application research of IFS, and the model of IFSP is also useful to intuitionistic fuzzy reasoning. Furthermore, this method can be generalized to interval-valued intuitionistic fuzzy sets as in \([18, 23]\).

II. CONSTRUCTION OF IFSP

Definition 1. An IFS \( A \) in universe \( X \) is given as follows (Atanassov [1, 2]):

\[
A = \{ x, \mu_{A}(x), \nu_{A}(x), \pi_{A}(x) : x \in X \}. \tag{1}
\]

Where \( \mu_{A} : X \rightarrow [0, 1], \nu_{A} : X \rightarrow [0, 1] \) with the condition \( 0 \leq \mu_{A}(x) + \nu_{A}(x) \leq 1 \) for each \( x \in X \). The numbers \( \mu_{A}(x), \nu_{A}(x) \in [0, 1] \) denote membership function and non-membership function of \( x \) to \( A \), respectively. For each IFS in \( X \), we call \( \pi_{A}(x) = 1 - \mu_{A}(x) - \nu_{A}(x) \) hesitancy function of \( x \) to \( A \), \( 0 \leq \pi_{A}(x) \leq 1 \) for each \( x \in X \).

Definition 2. Let \( X \) be a universe of discourse. Being the expansion of IFS \( A \), an IFSDP \( A' \) in \( X \) is an object having the form: \( A' = \{ x, \mu'_{A}(x) = \mu_{A}(x), \nu'_{A}(x) = \nu_{A}(x), \pi'_{A}(x) = \pi_{A}(x), \} \). Let \( \mu'_{A}(x) = \mu_{A}(x) + \alpha_{A}(x), \nu'_{A}(x) = \nu_{A}(x) + \beta_{A}(x), \pi'_{A}(x) = \pi_{A}(x) + \gamma_{A}(x), \) where \( \mu_{A}(x), \nu_{A}(x), \pi_{A}(x) \) are the same as definition 1. And we have: \( \alpha_{A}(x) + \beta_{A}(x) + \gamma_{A}(x) = \pi_{A}(x) + \pi'_{A}(x), \)

\[
\alpha_{A}(x) \geq 0, \beta_{A}(x) \geq 0, \gamma_{A}(x) \geq 0, \mu'_{A}(x) = \mu_{A}(x) + \alpha_{A}(x), \nu'_{A}(x) = \nu_{A}(x) + \beta_{A}(x), \pi'_{A}(x) = \pi_{A}(x) + \gamma_{A}(x), \]

\[
\mu'_{A}(x) + \nu'_{A}(x) + \pi'_{A}(x) = \mu_{A}(x) + \nu_{A}(x) + \pi_{A}(x) + \alpha_{A}(x) + \beta_{A}(x) + \gamma_{A}(x) = \mu_{A}(x) + \nu_{A}(x) + \pi_{A}(x) = 1. \]

Thus, IFS is a special case of IFSDP when \( \alpha_{A}(x) = \beta_{A}(x) = 0 \).

Theorem 1. Let \( A' \) be an IFSDP as definition 2, then

\[
\mu'_{A}(x) - \mu_{A}(x) + \nu'_{A}(x) - \nu_{A}(x) + \pi'_{A}(x) - \pi_{A}(x) = \pi_{A}(x) - \pi'_{A}(x). \tag{2}
\]

From definition 2, we have formula (2).

According to definition 1, let all sample data be divided into three parts, \( \mu_{A}(x) \) being the firm support party of event \( A, \nu_{A}(x) \) representing the firm opposition party of event \( A, \) and \( \pi_{A}(x) \) showing all the absent party.

Thus, IFS is a special case of IFSDP as definition 2. If the proportion of the absent party converted to the support party is \( \lambda_{A}(x) \) and that converted to the opposition party is \( 1 - \lambda_{A}(x) \), the model will become intuitionistic fuzzy sets with single parameter, where

\[
\alpha_{A}(x) = \lambda_{A}(x)(\pi_{A}(x) - \pi'_{A}(x)),
\]

\[
\beta_{A}(x) = (1 - \lambda_{A}(x))\pi_{A}(x) - \pi'_{A}(x).
\]

If the firm absent party is \( \pi'_{A}(x) = (1 - \lambda_{A}(x))\pi_{A}(x) \), then \( \pi_{A}(x) - \pi'_{A}(x) = \lambda_{A}(x)\pi_{A}(x) \), and then we will obtain the other IFSP definition as follows:

Definition 3. An IFSDP \( A' \) which is the expansion of IFS \( A \) in universe \( X \) is given as follows:

\( A = \{ x, \mu'_{A}(x), \nu'_{A}(x), x \in X \}. \)

Where \( \mu'_{A}(x), \nu'_{A}(x), \pi'_{A}(x) \) represent membership function, non-membership function, and hesitancy function of \( x \) to \( A \), respectively. And we have:

\[
\mu'_{A}(x) = \mu_{A}(x) + \lambda_{A}(x)\lambda_{A}(x)\pi_{A}(x),
\]

\[
\nu'_{A}(x) = \nu_{A}(x) + \lambda_{A}(x)(1 - \lambda_{A}(x))\pi_{A}(x),
\]

\[
\pi'_{A}(x) = (1 - \lambda_{A}(x))\pi_{A}(x).
\]

Where \( 0 \leq \lambda_{A}(x) \leq 1 \), \( \lambda_{A}(x) = 0, 1 \), and \( \mu_{A}(x), \nu_{A}(x), \pi_{A}(x) \) are the same as definition 1.

When \( \lambda_{A}(x) \) and \( \lambda_{A}(x) \) are random variables, the IFSDP model becomes a random IFSDP model, and they possess the corresponding probability distributions. For example, if \( P \) is a probability function and we have:

If \( P(\lambda_{A}(x) = 0) = 1 \), then

\[
\mu'_{A}(x) = \mu_{A}(x), \nu'_{A}(x) = \nu_{A}(x), \pi'_{A}(x) = \pi_{A}(x).
\]

And if \( P(\lambda_{A}(x) = \lambda_{A}(x) = 1) = 1 \), then

\[
\mu'_{A}(x) = \mu_{A}(x) + \lambda_{A}(x)\pi_{A}(x),
\]

\[
\nu'_{A}(x) = \nu_{A}(x) + \lambda_{A}(x)(1 - \lambda_{A}(x))\pi_{A}(x),
\]

\[
\pi'_{A}(x) = (1 - \lambda_{A}(x))\pi_{A}(x).
\]

It is clear that we will get the following conclusions: when \( \lambda_{A}(x) = 0 \), IFSDP is IFS as definition 1; when
\(\lambda_{a0}(x) = 1\), IFSDP is fuzzy sets; when 0 < \(\lambda_{a0}(x) < 1\), and \(\lambda_{a1}(x) = 0\) or \(\lambda_{a1}(x) = 1\), it is a severe skewnessness IFS; when 0 < \(\lambda_{a0}(x) < 1\), and \(\lambda_{a1}(x) = 0.5\), IFSDP is a compromising one. If \(\lambda_{a1}(x) = \lambda_{a}, i = 0, 1\), and \(\lambda_{a}\) is constant, then IFSDP is an intuitionistic fuzzy sets with fixed double parameters, otherwise it is a variable model.

**III. Weighted Scoring Function**

Suppose \(T\) and \(F\) are two types of extreme IFS in \(X\), where \(T = \{x, 1, 0\} \times x \in X\) means \(\mu_{T}(x) = 1\) and \(\nu_{T}(x) = 0\), and \(F = \{x, 0, 1\} \times x \in X\) means \(\mu_{F}(x) = 0\) and \(\nu_{F}(x) = 1\).

We denote \(S_{a} = \frac{m(A, F)}{m(A, F) + m(A, T)}\) to be a weighted scoring function, where \(m (A, T)\) and \(m (A, F)\) denote distance measures. In this paper, the following weighted scoring function based on IFS will be used in the simulation.

\[
S_{a} = \frac{\sum_{x \in X} w_{x} [\mu_{x}(x) + \nu_{x}(x)] - 1}{\sum_{x \in X} w_{x} [\mu_{x}(x)]^{2} + \nu_{x}(x)^{2} + \pi_{x}(x)^{2}}
\]

Then we have the following formula (4):

\[
S_{a} = 0.5 + \frac{1}{4} \sum_{x \in X} w_{x} [\mu_{x}(x) - \nu_{x}(x)]
\]

Obviously, we have \(0 \leq S_{a} \leq 1\). If \(A = F\), then \(S_{a} = 0\), and if \(A = T\), then \(S_{a} = 1\). Since \(F\) indicates that all the example data is the firm opposition party of event \(A\), we define \(S_{a} = 0\), which means that the score of \(F\) decision is zero and the result of \(F\) cannot be selected. Similarly, we define \(S_{a} = 1\), which means that the result of \(T\) is perfect and should be selected. Suppose that \(A^{*}\) is an IFSDP being the expansion of IFS \(A\), and then we can define its weighted scoring function as follows:

\[
S'_{a} = \frac{m(A', F)}{m(A', F) + m(A', T)}
\]

\[
U = \sum_{x \in X} w_{x} [\mu_{x}(x) - \nu_{x}(x) + 2 \pi_{x}(x) \lambda_{a0}(x) \lambda_{a1}(x)]
\]

\[
L = \sum_{x \in X} w_{x} [4 \mu_{x}(x)^{2} + \nu_{x}(x)^{2} + \mu_{x}(x) \nu_{x}(x) - 1.5 \mu_{x}(x) - 1.5 \nu_{x}(x) + 1]
\]

\[
U + 4 \pi_{x}(x) \lambda_{a0}(x) \lambda_{a1}(x) - \lambda_{a1}(x) + 1
\]

\[
+ 4 \pi_{x}(x) (\mu_{x}(x) - \nu_{x}(x)) \lambda_{a0}(x) \lambda_{a1}(x)
\]

\[
+ (4 \mu_{x}(x) + 8 \nu_{x}(x) - 6) \pi_{x}(x) \lambda_{a0}(x) \lambda_{a1}(x)
\]

According to formula (5, 6), we can draw a conclusion that \(S_{a}\) is a special case of \(S'_{a}\) when \(\lambda_{a0}(x) = 0\), and that they both have some similar properties as follows.

**Definition 4.** Let \(A\) and \(B\) be two IFSs in universe \(X\), \(A \subseteq B\) if and only if \(\mu_{A}(x) \leq \mu_{B}(x)\) and \(\nu_{A}(x) \leq \nu_{B}(x)\), for each \(x \in X\). Similarly, suppose that \(A^{*}\) and \(B^{*}\) are two IFSDPs as definition 2 (or definition 3), we also have \(A^{*} \subseteq B^{*}\) if and only if \(\mu_{A}^{*}(x) \leq \mu_{B}^{*}(x)\) and \(\nu_{A}^{*}(x) \leq \nu_{B}^{*}(x)\), for each \(x \in X\).

**Theorem 2.** Let \(A\) and \(B\) be two IFSs, and then we have: If \(A \subseteq B\), then \(\text{Score}(A) \leq \text{Score}(B)\). Similarly, let \(A^{*}\) and \(B^{*}\) be two IFSDPs, and then we have: If \(A^{*} \subseteq B^{*}\), then \(\text{Score}(A^{*}) \leq \text{Score}(B^{*})\).

**Proof:** According to formula (4), \(A \subseteq B\), thus we have \(\mu_{A}(x) \leq \mu_{B}(x)\), \(\nu_{A}(x) \leq \nu_{B}(x)\), and then we have \(\text{Score}(A) \leq \text{Score}(B)\). We can also obtain the same conclusion for IFSDPs.

**Definition 5.** Let \(A\) be an IFS in universe \(X\), \(A = \{x, \mu_{A}(x), \nu_{A}(x) | x \in X\}\), the complement of \(A\) is defined by: \(A^{c} = \{x, 1 - \mu_{A}(x), 1 - \nu_{A}(x) | x \in X\}\), for each \(x \in X\). Similarly, suppose that \(A^{c} = \{x, \mu_{A}(x), \nu_{A}(x) | x \in X\}\) is an IFSDP which is the expansion of IFS \(A\), and then the complement of \(A^{c}\) is defined by: \((A^{c})^{c} = \{x, \mu_{A}(x), \nu_{A}(x) | x \in X\}\).

**Theorem 3.** Let \(A\) be an IFS as mentioned above, then \(\text{Score}(A) + \text{Score}((A^{c})^{c}) = 1\) for each IFSDP \(A^{c}\) which is the expansion of IFS \(A\).

**Proof:**

\[
S_{a} = 0.5 + \frac{1}{4} \sum_{x \in X} w_{x} [\mu_{x}(x) - \nu_{x}(x)]
\]

\[
S_{c} = 0.5 + \frac{1}{4} \sum_{x \in X} w_{x} [\nu_{x}(x) - \mu_{x}(x)]
\]

We can also obtain the same conclusion for IFSDP \(A^{c}\). According to theorem 3 and formula (4), it is easy to get theorem 4.

**Theorem 4.** Let \(A\) be an IFS in universe \(X\), then:

\[
\text{Score}(A) = 0.5 \text{ if and only if } \mu_{A}(x) = \nu_{A}(x),
\]

\[
\text{Score}(A) < 0.5 \text{ if and only if } \mu_{A}(x) < \nu_{A}(x),
\]

and \(\text{Score}(A) > 0.5 \text{ if and only if } \mu_{A}(x) > \nu_{A}(x)\).

Similarly, we have the same property for each IFSDP \(A^{c}\) in universe \(X\) too.

From the definition of the random IFSDP model, we can construct the random scoring function model. For example: If \(P(\lambda_{a0}(x) = 0) = 1\), then \(S_{a} = S_{a}^{*}\).

Assume that \(P\) is a probability function, \(\lambda_{a0}(x)\) and \(\lambda_{a1}(x)\) are both random variables. Let
\[ \lambda_{d0}(x) = \lambda_{a0} \lambda_{d1}(x) = \lambda_{a1} . \] If the joint probability distribution \( P(\lambda_{d0}, \lambda_{d1}) \) is a continuous distribution for each \( \lambda_{d0} \in [0,1] \) and for each \( \lambda_{d1} \in [0,1] \), then we obtain the expectation of all the random decision-makings as follows:

\[ E_{\lambda_d} = \int_0^1 \int_0^1 S_d(\lambda_{d0}, \lambda_{d1}) d\lambda_{d0} d\lambda_{d1}. \]  

If we know the marginal probability distribution of \( (\lambda_{d0}, \lambda_{d1}) \), then we also have the marginal expectation of all the random decision-makings as follows:

\[ E_{\lambda_d} = \int_0^1 S_d(\lambda_{d0}, \lambda_{d1}) d\lambda_{d0} . \]  

IV. APPLICATION TO SINGLE ATTRIBUTE DECISION MAKING

In the following, we will apply the scoring function of IFS and IFSDP above to decision making. Suppose that \( A \) is feasible options set, \( A = \{ A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8 \} \). According to the following data from \( A \), we will make a choice among \( A \).

Assume that \( A(i) = \{ < x, 0.3 \}, A_2 = \{ < x, 0.6, 0.35 >, A_3 = \{ < x, 0.6, 0.4 >, A_4 = \{ < x, 0.5, 0.4 >, A_5 = \{ < x, 0.4, 0.4 >, A_6 = \{ < x, 0.4, 0.5 >, A_7 = \{ < x, 0.3, 0.7 >, A_8 = \{ < x, \mu_{d1}(x), \nu_{d1}(x) > | x \in X \}. \] For example, \( A_1 \) means that the degree of membership is \( \mu_{d1}(x) = 0.7 \), and that the degree of non-membership is \( \nu_{d1}(x) = 0.3 \), thus, the degree of hesitancy is \( \pi_{d1}(x) = 0 \) for \( x \in X \). According to their practical significance, we have:

\[ A_3 \subseteq A_4 \subseteq A_5 \subseteq A_6 \subseteq A_7 \subseteq A_1 \subseteq A_2 \subseteq A_8 . \]

In the following, we will compare the results calculated by conventional ranking functions of IFS with the results calculated by the scoring function of IFS and IFSDP.

According to the practical significance of IFS, we can make decision applying the degree of membership. Thus, we have the following membership function [(11)]: For each \( x \in X \),

\[ R_{\mu}(A) = \sum_{x \in X} w(x) \mu(x), i = 1, 2, ..., n. \]  

Chen and Tan proposed the following ranking function to make decision ([2]): For each \( x \in X \),

\[ R_{\mu} \left( A_i \right) = \sum_{x \in X} w(x) (\mu(x) - \nu(x)), i = 1, 2, ..., n. \]  

Hong and Choi introduced the following ranking function to make decision ([3]): For each \( x \in X \),

\[ R_{\mu} \left( A_i \right) = \sum_{x \in X} w(x) (\mu(x) + \nu(x)), i = 1, 2, ..., n. \]  

From formulas (4, 9, 10, 11), we can get the results as Figure 1. For example, \( R_{\mu}(A), R_{\mu}(A_1), R_{\mu}(A_2), \) and \( S_x \) are calculated by formulas (4, 9, 10, 11), and then we have:

\[ R_{\mu}(A) = 0.7, R_{\mu}(A_1) = 0.4, R_{\mu}(A_2) = 0.9. \]

According to Xu in [11], let \( m(A_i, A_j) = \sum_{x \in X} w(x) \left[ \max(\mu(x), \mu(x)) + |\nu(x) - \nu(x)| \right] \) for \( i \neq j \), 

\[ m(A_i, A_j) = \sum_{x \in X} w(x) \left[ \max(\mu(x), \mu(x)) + |\nu(x) - \nu(x)| \right] \]  

where \( A \) is a continuous distribution for \( 0 \leq x \leq 1 \). For example,

\[ R_w(A_i) = \frac{m(A_i, A_i)}{m(A_i, A_i) + m(A_i, A_j)} i = 1, 2, ..., n. \]  

Xu made use of four distance measures and applied them to formula (12) to make decision.

According to Xu in [11], let \( p = 2 \), calculated by Xu’s formulas, and then we have:

\[ m(A_i, A_j) = \sum_{x \in X} w(x) \left[ \max(\mu(x), \mu(x)) + |\nu(x) - \nu(x)| \right] \]  

From formulas (4, 9, 10, 11), we can get the results as Figure 1. For example, \( R(x), R_{\mu}(A), R_{\mu}(A_1), \) and \( S_x \) are calculated by formulas (4, 9, 10, 11), and then we have:

\[ R_{\mu}(A) = 0.7, R_{\mu}(A_1) = 0.4, R_{\mu}(A_2) = 0.9. \]  

We define:

\[ S_x = \frac{\mu_d(x)^2 + (\nu_d(x) - 1)^2 + \pi_d(x)^2}{0.7 \times 0.7 + 0.7 \times 0.7 + 0.3 \times 0.3 + 0.3 \times 0.3 + 0.3 \times 0.3} = 0.8448. \]
Let \( A \) be the set of alternatives, and \( A^* = \{ x \in \mathbb{R} | x \geq 0.3, 0.7 > x < 1 \} \). For example, \( A = \{ x | x \in [0, 1, 0, 2, 0.3] \} \).

From formula (13), we have Figure 2. For example, if \( A^* = \{ x \in \mathbb{R} | x \geq 0.3, 0.7 > x < 1 \} \), \( A^* = \{ x | x \in [0, 1, 0, 2, 0.3] \} \), then we have:

\[
m(A', A) = \frac{m(A', A)}{m(A, A) + m(A', A)}
\]

Similarly, we can obtain other results as in Figure 2.

According to the ranking of \( A_9 \), Xu’s methods, Chen’s method and scoring function method are more effective than Hong’s method and membership method, and they are approximate on the final decision results, which is: \( A_9 < A_8 < A_7 < A_6 < A_5 < A_4 < A_3 < A_2 < A_1 \). However, there are some differences among Xu’s methods, Chen’s method and scoring function method, and the difference among the slopes of them are the most significant difference. Obviously, the slopes of them are ranked as follows: Chen’s method > scoring function method > Xu3 > Xu1 = Xu2 > X4.

From formula (6), we get the following formulas:

\[
S_{A_{k}}^* = 0.5 + \frac{0.4 \lambda_{A_{k}}(x) \lambda_{A_{k}}(x) - 0.2 \lambda_{A_{k}}(x)}{0.16 A_{k}(A \lambda_{A_{k}}(x) - A_{k}(x) + 1) - 0.24 A_{k}(x) + 1.12}, \quad k = 1, 2, 3, 4, 5.
\]

And then we have the following Figure 3. From Figure 3, according to IFSDP, we have the same conclusion:

\[
A_9 < A_8 < A_7 < A_6 < A_5 < A_4 < A_3 < A_2 < A_1.
\]

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From formula (6), we get the following formulas:

\[
S_{A_{k}}^* = 0.5 + \frac{0.4 \lambda_{A_{k}}(x) \lambda_{A_{k}}(x) - 0.2 \lambda_{A_{k}}(x)}{0.16 A_{k}(A \lambda_{A_{k}}(x) - A_{k}(x) + 1) - 0.24 A_{k}(x) + 1.12}, \quad k = 1, 2, 3, 4, 5.
\]

And then we have the following Figure 3. From Figure 3, according to IFSDP, we have the same conclusion:

\[
A_9 < A_8 < A_7 < A_6 < A_5 < A_4 < A_3 < A_2 < A_1.
\]

Similarly, we can obtain other results as in Figure 2.

According to the ranking of \( A_9 \), Xu’s methods, Chen’s method and scoring function method are more effective than Hong’s method and membership method, and they are approximate on the final decision results, which is: \( A_9 < A_8 < A_7 < A_6 < A_5 < A_4 < A_3 < A_2 < A_1 \). However, there are some differences among Xu’s methods, Chen’s method and scoring function method, and the difference among the slopes of them are the most significant difference. Obviously, the slopes of them are ranked as follows: Chen’s method > scoring function method > Xu3 > Xu1 = Xu2 > X4.

From formula (6), we get the following formulas:

\[
S_{A_{k}}^* = 0.5 + \frac{0.4 \lambda_{A_{k}}(x) \lambda_{A_{k}}(x) - 0.2 \lambda_{A_{k}}(x)}{0.16 A_{k}(A \lambda_{A_{k}}(x) - A_{k}(x) + 1) - 0.24 A_{k}(x) + 1.12}, \quad k = 1, 2, 3, 4, 5.
\]
offers five feasible options $A_i$ $(i=1, 2, 3, 4, 5)$, which might be adapted to the physical structure of the library. Suppose that three attributes $C_j$ (economic, $C_A$ (functional), and $C_O$ (operational)) are taken into consideration in the installation problem, the weight vector of the attributes $C_j$ $(j=1,2,3)$ is $w=(0.3, 0.5, 0.2)^T$. Assume that the characteristics of the options $A_i$ $(i=1,2,3,4,5)$ are represented by IFS, shown as follows:

$A_1 = \{<C_1, 0.2, 0.4>, <C_2, 0.7, 0.1>, <C_3, 0.6, 0.3>\}$,
$A_2 = \{<C_1, 0.4, 0.2>, <C_2, 0.5, 0.2>, <C_3, 0.8, 0.1>\}$,
$A_3 = \{<C_1, 0.5, 0.4>, <C_2, 0.6, 0.2>, <C_3, 0.9, 0>\}$,
$A_4 = \{<C_1, 0.3, 0.5>, <C_2, 0.8, 0.1>, <C_3, 0.7, 0.2>\}$,
$A_5 = \{<C_1, 0.8, 0.2>, <C_2, 0.7, 0.1>, <C_3, 0.1, 0.6>\}$.

We will compare the results calculated by conventional distance measures of IFS ([11]) with the results calculated by the score function of IFS and IFSDP. Calculated by formulas (4, 6) above, we can obtain the results as follows. We only calculate the results using the score function, and the results of Xu’s methods are from [11]. For example:

By formula (4), we calculate the following scores:

\[ S_{A_1} = 0.5 + 0.3 = 0.7113, \]
\[ S_{A_2} = 0.5 + 0.41 = 0.8106, \]
\[ S_{A_3} = 0.5 + 0.43 = 0.7701. \]

Similarly, we have the following results in Table 1.

From formula (4), we have $A_1 > A_2 > A_3 > A_4 > A_5$, and this result is the same as that in [11]. From Table 1, we have the optimal decisions in Table 2.

**TABLE I. MULTIPLE ATTRIBUTE DECISION MAKING BASED ON RANKING FUNCTION OF IFS**

<table>
<thead>
<tr>
<th>Ranking function</th>
<th>$Xu_1$</th>
<th>$Xu_2$</th>
<th>$Xu_3$</th>
<th>$Xu_4$</th>
<th>Scoring function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.4685</td>
<td>0.4813</td>
<td>0.4679</td>
<td>0.4591</td>
<td>0.7113</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.5083</td>
<td>0.5315</td>
<td>0.4679</td>
<td>0.4835</td>
<td>0.7431</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.5555</td>
<td>0.5793</td>
<td>0.5833</td>
<td>0.5889</td>
<td>0.8106</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.5147</td>
<td>0.5313</td>
<td>0.551</td>
<td>0.5347</td>
<td>0.7859</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.4954</td>
<td>0.513</td>
<td>0.5372</td>
<td>0.5331</td>
<td>0.7701</td>
</tr>
</tbody>
</table>

From definition 3, we will apply the scoring function of IFSDP to multiple attribution decision making. According to definition 3, we have formula (6), and then we have the following formulas:

\[ S_{A_1} = 0.1425(x_1 + \lambda) - 0.1429(x_1 + 1), \]
\[ S_{A_2} = 0.2895(x_2 + \lambda) - 0.2899(x_2 + 1), \]
\[ S_{A_3} = 0.3825(x_3 + \lambda) - 0.383(x_3 + 1), \]
\[ S_{A_4} = 0.525(x_4 + \lambda) - 0.5252(x_4 + 1), \]
\[ S_{A_5} = 0.7859(x_5 + \lambda) - 0.7859(x_5 + 1). \]

According to the formulas $S_{A_1}, S_{A_2}, S_{A_3}, S_{A_4}$ and $S_{A_5}$, we describe the three-dimensional space figure (Figure 5) and the section figure (Figure 6) when $\lambda_{A_1}(x) = 0.5$. Where we note $\lambda_{A_1}(x) = \lambda_{A_1}(x)$. From Figure 5, $A_1$ is the optimal decision in most cases only except the right upper corner, where $\lambda_{A_1}(x) \rightarrow 1$ and $\lambda_{A_1}(x) \rightarrow -1$. According to definition 3, $\lambda_{A_0}(x) \rightarrow -1$ indicates that the proportion of the absent party being converted into another party is high, and $\lambda_{A_1}(x) \rightarrow 1$ indicates that the proportion of the convertible absent party being converted into the support party is high. Thus, we know that when the majority of the absent party can be converted and most of them are converted into the support party, $A_1$ is the optimal decision. Based on the definitions of $A_2$ and $A_3$, $A_2 = \{<C_1, 0.4, 0.2>, <C_2, 0.5, 0.2>, <C_3, 0.8, 0.1>\}$, and $A_3 = \{<C_1, 0.5, 0.4>, <C_2, 0.6, 0.2>, <C_3, 0.9, 0>\}$. There are only slight differences between $A_2$ and $A_3$. We have $A_2 > A_3$ for attribute $C_2$ and attribute $C_3$, while it is difficult to judge which one is the better between $A_2$ and $A_3$ for attribute $C_1$. The hesitancy degree of $A_2$ is bigger than that of $A_3$, thus, $A_2$ is impacted more than $A_3$ by the extreme variation of the absent party in decision-making, and is less stable. If most of the absent party is converted into the support party, $A_2 > A_3$; otherwise, $A_2 > A_3$. Considering that $A_2$ is more stable than $A_3$, $A_2$ should be the optimal decision in the actual decision-making.

From Figure 6, the known condition is $\lambda_{A_1}(x) = 0.5$, which means that half of the convertible absent party is converted into the support party, and the other is...
converted into the opposition party. Then we have the following results:

Figure 6: Section figure if $\lambda_{d}(x) = 0.5$.

If $\lambda_{d}(x) < 0.45$, then $A_1 > A_2 > A_3 > A_4$, and this result is the same as IFS.

If $0.45 < \lambda_{d}(x) < 0.73$, then $A_1 > A_2 > A_3 > A_4 > A_5$.

If $\lambda_{d}(x) > 0.73$, then $A_1 > A_2 > A_3 > A_4 > A_5$.

Compared with these results, $A_3$ is the optimal decision for $\lambda_{d}(x) \in [0, 1]$. Similarly, we can also study the variation of $A_i$ with $\lambda_{d}(x)$ when $\lambda_{d}(x)$ is known.

If $\lambda_{d}(x) = 0$, then IFSP is equivalent to IFS, and then we have $\mu_i(x) = \mu_i(x), v_i(x) = v_i(x), \pi_i(x) = \pi_i(x)$, which means that the result of the left plane in Figure 5 is the result calculated by IFS. Similarly, the result of the left coordinate axis in Figure 6 is also the result calculated by IFS.

In [11], professor Xu applies four kinds of distance measures to make decisions. In this paper, we use only one scoring function to make decisions, and the results are the same as Xu’s. Furthermore, by analyzing the variation of the indeterminacy degree, we reveal a potential decision making result $A_2$ and the reason for selecting $A_2$. Because it is the first time for us to explore the application of IFSP to multiple attribute decision making, we assume that $\lambda_{d}(x)$ is the same value for all the attributes, and $\lambda_{d}(x)$ is similar to $\lambda_{d}(x)$. In practical decision-making, since the proportion of the convertible absent party may be different for all the attributes, researcher can set more parameters to meet the needs of actual problem in reality.

The experiment results above show that there is much difference between multiple attribute decision making results of IFSP and that of IFS. Conventional IFS method is simple, but its decision making results are fixed when it is calculated by conventional ranking functions. Thus, it is difficult to reveal the potential law from all the available information when using the IFS method. And the results of the IFSP method are variable, which can be adjusted to possible results with the variation of the parameters. Furthermore, for supervised models, if the fixed decision making results of IFS are different from the results of the practical data and the actual decisions, the IFS method will fall into fail. However, when using the IFSP method, we can meet the needs of the practical data and the actual decision by adjusting the parameters to appropriate values. All the results above show that the IFSP method is more comprehensive and flexible than the IFS method.

**VIII. EXPECTATION SCORING FUNCTION APPLIED TO MULTIPLE ATTRIBUTE DECISION MAKING**

From formula (8), we obtain the marginal expectation ranking functions as follows:

$$E_{\lambda, A_{d}} S'_{A_{d}} \equiv \int_{\lambda_{d}} S'_{A_{d}} P(\lambda_{d}, 0) d\lambda_{d}.$$  \hspace{1cm} (14)

$$E_{\lambda, A_{d}} S'_{A_{d}} \equiv \int_{\lambda_{d}} S'_{A_{d}} P(\lambda_{d}, 0) d\lambda_{d}.$$  \hspace{1cm} (15)

Where $P(\lambda_{d}, 0)$ represents the joint probability distribution of the random vector $(\lambda_{d}, 0)$. If the joint probability distribution of $(\lambda_{d}, 0)$ is a continuous uniform distribution for each $\lambda_{d} \in [0, 1]$ and for each $0 \leq \lambda_{d} < 1$, then we obtain the formulas as follows:

$$E_{\lambda, A_{d}} S'_{A_{d}} \equiv \int_{\lambda_{d}} S'_{A_{d}} d\lambda_{d}.$$  \hspace{1cm} (16)

$$E_{\lambda, A_{d}} S'_{A_{d}} \equiv \int_{\lambda_{d}} S'_{A_{d}} d\lambda_{d}.$$  \hspace{1cm} (17)

Considering formula (6), we have:

$$S'_{A_{d}} = 0.5 + \frac{a_{1}\lambda_{d0} + a_{2}}{a_{1}\lambda_{d0} + a_{2} + b_{1}\lambda_{d1} + b_{2} + b_{3}}.$$  \hspace{1cm} (18)

$$S'_{A_{d}} = 0.5 + \frac{a_{1}\lambda_{d0} + a_{2}}{a_{1}\lambda_{d0} + a_{2} + b_{1}\lambda_{d1} + b_{2} + b_{3} + b_{3}}.$$  \hspace{1cm} (19)

Where $a_{1} = \sum_{i=1}^{n} w_{i}[2\pi_{i}(x)\lambda_{d1} - \pi_{i}(x)]$,
\[ E_{\lambda} S_{\lambda} = 0.5 + \int_{0}^{1} \frac{b_{1} \lambda_{\beta} + b_{3}}{b_{1} \lambda_{\beta} + b_{2} \lambda_{\beta} + b_{3}} d \lambda_{\beta}. \quad (21) \]

Obviously, formula (20) and formula (21) can be calculated from the integral method of rational function.

IX. CONCLUSION

We propose a novel IFSP model according to IFS and apply it to multiple attribute decision making. The IFS method not only involves membership function and non-membership function, but also involves the detachment of hesitancy function. Therefore, it is more effective than the IFS method.

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