Integrability of the Reduction Fourth-Order Eigenvalue Problem

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Abstract—To study the reduced fourth-order eigenvalue problem, the Bargmann constraint of this problem has been given, and the associated Lax pairs have been nonlineared. By means of the viewpoint of Hamilton mechanics, the Euler-Lagrange function and the Legendre transformations have been derived, and a reasonable Jacobi-Ostrogradsky coordinate system has been found. Then, the Hamiltonian canonical coordinate system equivalent to this eigenvalue problem has been obtained on the symplectic manifolds. It is proved to be an infinite-dimensional integrable Hamilton system in the Liouville sense. Moreover the involutive representation of the solutions is generated for the evolution equations hierarchy in correspondence with this reduced fourth-order eigenvalue problem.

Index Terms—constraint flow, Bargmann system, integrable system, involutive representation

I. INTRODUCTION

The theory of integrable systems has been an interesting and important problem. The finite dimensional integrable system is used to describe the problems in mathematical physics, mechanics, etc., such as Kovalevskia top, geodesic flows on the ellipsoid harmonic oscillator equation on sphere, Calogero-Moser system, etc [1-3]. The soliton equation as an infinite dimensional integrable system is one of the most prominent subjects in the field for the nonlinear science. “Solutions” were first observed by J. Scott Russell in 1834 whilst riding on horseback beside the narrow Union canal near Edinburgh, Scotland. There are a number of discussions in the literature describing Russell’s observations. At the center of these observations is the discovery that these nonlinear waves can interact elastically and continue afterward almost as if there had been no interaction at all (see Fig. 1- Fig. 5). Because of the analogy with particles, Zabusky and Kruskal named these special waves as solitons. Zabusky and Kruskal’s remarkable numerical discovery demanded an analytical explanation and detailed mathematical study of the partial differential equations. In 1968, Lax put the inverse scattering method for solving the KDV equation into a more general framework which subsequently paved the way to generalizations of the technique as a method for solving other partial differential equations [1-4]. The technique of the so-called nonlinearization of Lax pairs [5,8-10] has been developed and applied to various soliton hierarchies, from which a large of interesting finite-dimensional Liouville integrable Hamiltonian systems have been obtained. Recently, this method was generalized to discuss the nonlinearization of Lax pairs and adjoint Lax pairs of soliton hierarchies [8-11]. The nonlinearization approach (or constrained flows) of eigenvalue problems or Lax Pairs has been used to seek the relations between the infinite and finite integrable system. There are series of finite integrable system obtained by this approach [8-19].
The fourth-order eigenvalue problem
\[ L\phi = \left( \partial^4 + q \partial^2 + p \partial + r \right) \phi = \lambda \phi \]
has been discussed, but its reduced system
\[ L\phi = \left( \partial^4 + \partial u \partial + v \right) \phi = \lambda \phi \]
In this paper, we consider the reduced fourth-order eigenvalue problem
\[ L\phi = \left( \partial^4 + \partial u \partial + v \right) \phi = \lambda \phi \]
Here, \( \partial = \frac{\partial}{\partial x} \), eigenparameter \( \lambda \in \mathbb{R} \), and potential function \( u, v \) is a actual function in \((x, t)\). By means of

the viewpoint of Hamilton mechanics, the Euler-Lagrange function and the Legendre transformations have been derived, and a reasonable Jacobi-Ostrogradsky coordinate system has been found. Then, the Hamiltonian canonical coordinate system equivalent to the eigenvalue problem (1) has been obtained on the symplectic manifold. It is proved to be an infinite-dimensional integrable Hamilton system in the Liouville sense. Moreover the involutive representation of the solutions is generated for the evolution equations hierarchy in correspondence with the reduced fourth-order eigenvalue problem (1).

II. MAIN RESULTS

A. Evolution Equations and Lax Pairs

Now, suppose \( \Omega \) is the basic interval of the reduced eigenvalue problem (1), if the potentials \( u, v, \phi \) in (1) and their derivates on \( x \) are all decay at infinity, then \( \Omega = (-\infty, +\infty) \); if they are all periodic \( T \) functions, then \( \Omega = [0, 2T] \).

Definition 1 [8, 9] Assume that our linear space is equipped with a \( L \) scalar product
\[ (f, g)_{l, (\Omega)} = \int_{\Omega} f^* g^* dx < \infty \]
the symbol * denotes the complex conjugate.

Definition 2 [8, 9] Operator \( A \) is called dual operator of \( A \), if
\[ (Af, g)_{l, (\Omega)} = (f, Ag)_{l, (\Omega)} \]
and \( A \) is called a self-adjoint operator, if
\[ A = A^* \]

We consider the following fourth-order operator \( L \) on the interval \( \Omega \)
\[ L = \partial^4 + \partial u \partial + v \]
here \( u, v \) is potential function of the eigenvalue problem (1). \( \lambda \) is called eigenvalue of the eigenvalue problem (1), and \( \phi \) is called eigenfunction to eigenvalue \( \lambda \), if
\[ L\phi = \lambda \phi \]
\[ \phi \in L_2(\Omega) \] is non-trivial solution of the eigenvalue problem (1).

Theorem 1 \( L = \partial^4 + \partial u \partial + v \) is a self-adjoint operator on \( \Omega \).

Proof. By Definition 2, we easily derive Theorem1.

If \( L' = \frac{\partial}{\partial \mu} \left[ L(u + \varepsilon \mu) \right] \) is called the derived function of differential operator \( L = L(u) \), here \( u \) is potential function of \( L \), then we have

Theorem 2 If \( \phi \) is an eigenfunction corresponding to the eigenvalue \( \lambda \) of (1), then functional Gradient [20]
\[ \nabla \lambda = \left( \frac{\partial \lambda}{\partial u}, \frac{\partial \lambda}{\partial \mu} \right) = \left( \int_{\Omega} \phi^2 dx \right)^{-1} \begin{pmatrix} \phi^2 \\ \phi^2 \end{pmatrix} \]

Proof. By (1), we have
\[ L'\phi + L\phi' = \lambda' \phi + \lambda \phi' \]
and
\[ \int_{\Omega} (L \phi') \phi dx = \int_{\Omega} \phi' (L \phi) dx = \int_{\Omega} \lambda \phi' \phi dx \]

then
\[ L \phi' = \lambda \phi' \]
hence
\[ L \phi = \lambda \phi \]

Namely:
\[ \int_{\Omega} \lambda \phi' \phi dx = \int_{\Omega} (L \phi') \phi dx \]
\[ = \int_{\Omega} \left[ (\delta u \phi'^2 + \delta u \phi' \delta u') \phi \right] dx \]
\[ = \int_{\Omega} \left[ \delta \phi \phi \phi + \delta u (\phi' \phi + \phi \phi') + \delta \phi \phi \phi \right] dx \]
\[ = \int_{\Omega} \left[ \delta \phi \phi \phi + \delta u \phi' \phi \right] dx \]

so
\[ V \lambda = \left( \frac{\partial \phi}{\partial \nu} \right) = \left( \int_{\Omega} \phi' \phi dx \right)^{-1} \left( -\phi' \phi \right) \]

**Definition 3** [21]. The commutator of operators \( W \) and \( L \) is defined as follows [21, 22].

Set
\[
W_n = \sum_{j=L}^{n} \left[ \begin{array}{c}
\frac{1}{4} b_j \phi_j - \frac{1}{8} b_j \phi_j^2 + \frac{1}{8} (2a_j + b_{jj} + 2ab_j) \phi_j \\
\frac{1}{8} (3a_j + b_{jj} + ub_j) \phi_j + \frac{1}{4} \phi_j - 1
\end{array} \right] L^{m-j}, \quad m = 0, 1, 2, \ldots
\]

\[ G_m = \left( \begin{array}{c}
\hat{a}_j \\
\hat{b}_j
\end{array} \right), \quad j = 0, 1, 2, \ldots, m \]

\[ J = \left( \begin{array}{c}
0 \\
\frac{3}{4} \phi \phi + \frac{1}{2} u \phi + \frac{1}{4} u
\end{array} \right) \]

\[ K = \left( \begin{array}{c}
K_{11} \\
K_{12} \\
K_{21} \\
K_{22}
\end{array} \right) \]

here
\[ K_{11} = 4 \frac{1}{5} \phi \phi + \frac{1}{2} u \phi + \frac{1}{4} u, \]
\[ K_{12} = \frac{3}{8} \phi \phi + \frac{3}{4} u \phi + \frac{3}{8} u + \frac{1}{4} v, \]
\[ K_{21} = \frac{3}{8} \phi \phi + \frac{3}{4} u \phi + \frac{3}{8} u + \frac{1}{4} v, \]
\[ K_{22} = \frac{1}{8} \phi \phi + \frac{1}{2} u \phi + \frac{1}{4} u + \frac{1}{4} v + \frac{1}{2} v + \frac{1}{4} v \]

**Theorem 3** Operators \( J \) and \( K \) are the bi-Hamilton operators [5, 8, 9], moreover if \( \lambda \) and \( \phi \) are the eigenvalue and eigenfunction of (1), then
\[ JG_{m+1} = KG_m \quad m = 0, 1, 2, \ldots \]

\[ KV \lambda = J \lambda V \lambda \]

Proof. By means of the Definition of the bi-Hamilton operators, it is easily derived that \( J \) and \( K \) are the bi-Hamilton operators by the definition of the bi-Hamilton operators.

By complex computing, we can obtain (6), (7).

Set \( JG'_m = 0 \), then
\[ G_0 = \left( \begin{array}{c}
\frac{4}{5} \\
0
\end{array} \right) \]

or
\[ G_0 = \left( \begin{array}{c}
-u \\
4
\end{array} \right) \]

So, we have
\[ L_n = [W_n, L] = \sum_{j=0}^{m} [L_n (KG_j) - L_n (JG_j)] L^{m-j} \]

\[ = L_n (KG_m) - L_n (JG_m) L^{m+1} \]

\[ = L_n (KG_m) = L_n (JG_m) \]

here, \( L_n (\zeta) = \frac{d}{d \zeta} \left( \left( \frac{q}{p} \right) + \zeta \right); R^2 \to R \) is one to one [5, 8-10].

By the Hamilton operators \( J \) and \( K \) (4), (5), we have:

**Theorem 4** Set \( JG'_m = 0 \), then the m-order evolution equation of (1) are:
\[ q_m = X_m = JG_{m+1} = KG_m \]

here \( q = (u, v)' \), \( m = 0, 1, 2, \ldots \), and (10) become the isospectral compatible condition of the following Lax pairs
\[ [L \phi = \lambda \phi] \]

\[ \phi_{m} = W_{m} \phi \]

here \( m = 0, 1, 2, \ldots \).

So, we have the first evolution equation
\[ u_0 = u \]

\[ v_0 = v \]

and its Lax pairs
\[ [L \phi = \lambda \phi] \]

\[ \phi_{m} = \phi \]

or the first evolution equation
\[ u_0 = -\frac{5}{4} u_{xxx} - \frac{3}{4} u_{xxx} + 3v \]

\[ v_0 = \frac{3}{8} u_{xxxx} - \frac{3}{8} u_{xxxx} - \frac{3}{8} u_{xxxx} + v_{xxx} + \frac{3}{4} v_{xxx} \]

and its Lax pairs
\[ [L \phi = \lambda \phi] \]

\[ \phi_{m} = -\frac{3}{4} u + \phi + 3 \phi \]

**Remark** 1 The condition (8) generates the Bargmann system for the fourth-order eigenvalue problem (1); the condition (9) generates the C.Neumann system for the fourth-order eigenvalue problem (1) [1, 3-5].
B. Jacobi-Ostrogradsky Coordinates and Hamilton Canonical Forms of Bargmann System

Now, suppose \( \{ \lambda_j, \phi_j \} (j = 1, 2, \ldots, N) \) are eigenvalues and eigenfunctions for the fourth-order eigenvalue problem (1), and \( \lambda_1 < \lambda_2 < \cdots < \lambda_N \), then

\[
L \Phi = (\partial^4 + \partial u + v) \Phi = \Lambda \Phi
\]

(16)

here, \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N) \). Suppose \( \Phi = \text{diag}(\phi_1, \phi_2, \ldots, \phi_N)^T \).

From (7), we have

\[
K \left( \sum_{j=1}^{N} \lambda_j \partial^2 \phi_j \right) = J \left( \sum_{j=1}^{N} \lambda_j \partial^2 \phi_j \right),
\]

here \( k = 0, 1, 2, \ldots \).

So

\[
K \left( -\left\{ \lambda^4 \Phi, \Phi \right\} \right) = J \left( -\left\{ \lambda^3 \Phi, \Phi \right\} \right)
\]

(17)

Here \( k = 0, 1, 2, \ldots \).

Set \( G_i = (4, 0)^T \), using (6), we have:

\[
G_i = \left( -\frac{3}{4} u_x - \frac{3}{8} v^2 + v \right) u
\]

Now we define the constrained system to the Bargmann system

\[
G_i = \left( -\frac{3}{4} u_x - \frac{3}{8} v^2 + v \right) \Phi
\]

(18)

then, we have the relation between the potential \( (u, v) \) and the eigenvector \( \Phi \) as follows

\[
\left( \begin{array}{c}
u \\v
\end{array} \right) = \left( \begin{array}{c}
\left\{ \Phi, \Phi \right\} \\
\frac{3}{2} \left\{ \Phi, \Phi \right\} + \frac{1}{2} \left\{ \Phi, \Phi \right\} + \frac{3}{8} \left\{ \Phi, \Phi \right\}^2
\end{array} \right)
\]

(19)

From (6), (17), (18) and (19), we have

\[
G_i = \left( -\frac{3}{4} u_x - \frac{3}{8} v^2 + v \right) \left( -\left\{ \Phi, \Phi \right\} \right)
\]

(20)

Based on the constrained system (19), the eigenvalue problem \( L \Phi = \Lambda \Phi \)

is equivalent to the following system

\[
\Phi_{x x x x} + \left( \Phi, \Phi \right) + 2 \left( \Phi, \Phi \right) = \Lambda \Phi
\]

(21)

Remark 2 we call the equation system (21) to be the Bargmann system for the fourth-order eigenvalue problem (1).

In order to obtain the Hamilton canonical forms which is equivalent to the Bargmann system (21), the Lagrange function \( \hat{I} \) [8-9] is defined as follows:

\[
\hat{I} = \int_0^1 Idx
\]

(22)

where,

\[
I = \frac{1}{2} \left( \Phi_x, \Phi_x \right) + \frac{1}{2} \left( \Phi, \Phi \right) + \frac{1}{4} \left( \Phi, \Phi \right)^2
\]

(23)

Theorem 5. The Bargmann system (21) for the fourth-order eigenvalue problem (1) is equivalent to the Euler-Lagrange equations

\[
\frac{\partial I}{\partial \Phi} = 0
\]

(24)

Proof. By (22), we have

\[
\frac{\partial I}{\partial \Phi} = \frac{\partial I}{\partial \Phi} + \frac{\partial I}{\partial \Phi} = -\Lambda \Phi + \frac{1}{8} \left( 3 \left( \Phi, \Phi \right)^2 + 4 \left( \Phi, \Phi \right) + 12 \left( \Phi, \Phi \right) \right) \Phi
\]

\[
+ 2 \left( \Phi, \Phi \right) + \left( \Phi, \Phi \right) \Phi + \Phi
\]

so

\[
\frac{\partial I}{\partial \Phi} = L \Phi - \Lambda \Phi = 0
\]

Set \( y_i = -\Phi, y_2 = \Phi, \) and \( h = \sum_{j=1}^{N} (y_j, z_j) - I \), our aims are to find the coordinates \( z_i, z_j \) and the Hamilton function \( h \), that satisfy the following Hamilton canonical equations

\[
\left\{ \begin{array}{l}
y_{j x} = \left( y_j, h \right) = \frac{\partial h}{\partial z_j} \\
z_{j x} = \left( z_j, h \right) = -\frac{\partial h}{\partial z_j}.
\end{array} \right.
\]

(25)

here \( j = 1, 2 \), the symbol \( \left\{ \begin{array}{l}
y_{j x}
\end{array} \right\} \) is the Poisson bracket in the symplectic space \( \left( \omega = \sum_{j=1}^{N} dy_j \wedge dz_j, R^{2N} \right) \), and the Poisson bracket of Hamilton functions \( F, H \) in the symplectic space is defined as follows [1, 3, 4, 5]

\[
\left\{ F, H \right\} = \sum_{j=1}^{N} \sum_{k=1}^{N} \left( \frac{\partial F}{\partial y_j} \frac{\partial H}{\partial z_k} - \frac{\partial F}{\partial z_k} \frac{\partial H}{\partial y_j} \right)
\]

By \( h = \sum_{j=1}^{N} (y_j, z_j) - I \), then

\[
dh = \sum_{j=1}^{N} \left( (y_j, dz_j) + (z_j, dy_j) \right) dt.
\]

On the other hand, \( h = h(y_j, z_j), (j = 1, 2) \), so

\[
dh = \sum_{j=1}^{N} \left( (y_j, dz_j) + (z_j, dy_j) \right) dt.
\]

so that
\[ dI = (z_1, dy_1) + (z_2, dy_2) + (z_3, dy_3) + (z_4, dy_4) \]
\[ = - (z_1, d\Phi_1) + (z_2, d\Phi_2) + (z_3, d\Phi_3) + (z_4, d\Phi_4) \]
\[ = (z_1, d\Phi_1) + (z_2, d\Phi_2) + (z_3, d\Phi_3) + (z_4, d\Phi_4) \]

we have the relations:
\[ z_1 = \Phi_m + \frac{1}{2} \Phi_1 + \frac{1}{2} \Phi_1 \]
\[ z_2 = \Phi_m + \frac{1}{4} \Phi_2 \]

**Theorem 6.** The Jacobi-Ostrogradsky coordinates are as follows
\[
y_1 = -\Phi
\]
\[
y_2 = \Phi
\]
\[ z_1 = \Phi_m + \frac{1}{2} \Phi_1, \quad z_2 = \Phi_m + \frac{1}{4} \Phi_2 \]
(26)

and the Bargmann system (21) for the fourth-order eigenvalue problem (1) is equivalent to the Hamilton canonical system
\[ y_m = \{y_j, h\} = \frac{\partial h}{\partial y_j}, \quad z_m = \{z_j, h\} = -\frac{\partial h}{\partial y_j} \]
(27)

here \( j = 1, 2 \), and the Hamilton function \( h \) is
\[ h = \frac{1}{2} \{\lambda y_1, y_1\} - \{y_2, z_2\} - \frac{1}{4} \{y_1, y_2\}^2 + \frac{1}{2} \{y_1, y_1\} y_1 z_2 + \frac{1}{2} \{y_2, y_2\} \]
(28)

**Remark 3.** Based on the Jacobi-Ostrogradsky coordinate system (26), the Bargmann constrained equations associated with the fourth-order eigenvalue problem (1) is
\[
u = \frac{3}{2} y_1 z_2 + \frac{3}{2} y_1 z_2 - \frac{3}{8} y_1 y_2 \]
\[ v = -\frac{3}{2} y_1 z_2 + \frac{1}{2} y_1 y_2 \]
(29)

**C. Hamilton Equations of Bargmann System**

**Theorem 7** The Bargmann system (21) for the fourth-order eigenvalue problem (1) is equivalent to:
\[ Y = MY \]
(30)

where,
\[ Y = (y_1, y_2, y_3, y_4) \]
\[ = (-\Phi_1, \Phi_2, \Phi_3, \Phi_4 + \frac{1}{2} \Phi_2, \Phi_3 + \frac{1}{4} \Phi_2, \Phi_4) \]
(31)

\[
M = \begin{pmatrix}
0 & -E & 0 & 0 \\
-\frac{1}{2} uE & 0 & E & 0 \\
-\frac{1}{4} uE & 0 & 0 & E \\
-\lambda + \left(\frac{v}{4} \right) & 0 & E & -\frac{1}{2} uE & 0 \\
\end{pmatrix}
\]

and the system of the Lax pairs for the evolution equation hierarchy (10) is equivalent to
\[ Y_m = W_m Y \]
(33)

where \( m = 0, 1, 2, \ldots \), and
\[ W_m = (\omega_n)^{n-4} \quad m = 0, 1, 2, \ldots \]
(34)

\[ \omega_1 = \sum_{j=1}^n \left( \frac{1}{8} b_{j-1, 1} + \frac{1}{16} u b_{j-1, 1} + \frac{1}{4} a_{j-1, 1} \right) \lambda^{m-j} \]
\[ \omega_2 = \sum_{j=1}^n \left( \frac{1}{8} b_{j-1, 1} + \frac{1}{4} u b_{j-1, 1} + \frac{3}{8} a_{j-1, 1} \right) \lambda^{m-j} \]
\[ \omega_3 = \sum_{j=1}^n \left( \frac{1}{8} b_{j-1, 1} \right) \lambda^{m-j} \]
\[ \omega_4 = \sum_{j=1}^n \left( \frac{1}{4} b_{j-1, 1} \right) \lambda^{m-j} \]

Here, \[ b_{j-1, 1} = 0, 1, 2, \ldots \]

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\[ \omega_{3m} = \sum_{j=0}^{\infty} \left[ \frac{1}{16} ub_{j-1x} - \frac{1}{8} a_{j-1x} \right] \Lambda^{m-j} \]
\[ \omega_{4m} = \sum_{j=0}^{\infty} \left[ -\frac{1}{8} b_{j-1x} - \frac{1}{8} ub_{j-1x} - \frac{1}{4} a_{j-1x} \right] \Lambda^{m-j} \]
\[ \omega_{4m} = \sum_{j=0}^{\infty} \left[ \frac{1}{32} ub_{j-1xxx} - \frac{1}{32} \Lambda^{m-j} \right] - \frac{5}{32} \Lambda^{m-j} \]
\[ -\left( \frac{3}{32} u_{ss} + \frac{1}{4} v \right) b_{j-1x} - \left( \frac{1}{32} u_{tt} + \frac{3}{16} v_{s} + \frac{1}{32} u_{ss} \right) b_{j-1x} \]
\[ -\left( \frac{1}{16} v_{s} + \frac{1}{64} u_{xx} \right) b_{j-1x} - \frac{1}{16} a_{j-1xxx} + \frac{1}{8} u_{jxxx} \]
\[ + \left( \frac{1}{8} v_{s} + \frac{1}{64} u_{xx} \right) \Lambda^{-j} \]
\[ + \sum_{j=0}^{\infty} \left[ \frac{1}{8} b_{j-1x} - \frac{1}{4} a_{j-1x} \right] \Lambda^{m-j} \]
\[ \omega_{2m} = \sum_{j=0}^{\infty} \left[ \frac{1}{16} u_{jxxx} + \frac{3}{32} u_{jxxx} + \left( -\frac{1}{32} u_{s} + \frac{1}{4} v_{s} + \frac{1}{8} u_{ss} \right) b_{j-1x} \right] \]
\[ + \left( \frac{1}{8} v_{s} - \frac{1}{64} u_{xx} \right) b_{j-1x} - \frac{1}{4} a_{j-1xxx} + \frac{1}{16} u_{jxxx} \]
\[ - \sum_{j=0}^{\infty} \left[ \frac{1}{8} b_{j-1x} \right] \Lambda^{m-j} \]
\[ \omega_{3m} = \sum_{j=0}^{\infty} \left[ \frac{1}{8} b_{j-1xxx} + \frac{5}{32} u_{jxxx} + \frac{1}{16} v_{jxxx} \right] \Lambda^{m-j} \]
\[ + \frac{3}{8} a_{j-1xxx} + \frac{1}{64} u_{jxxx} \right] \Lambda^{m-j} - \sum_{j=0}^{\infty} \left[ \frac{1}{4} b_{j-1x} \right] \Lambda^{m-j} \]
\[ \omega_{4m} = \sum_{j=0}^{\infty} \left[ -\frac{1}{16} ub_{j-1x} - \frac{1}{16} u_{jxxx} - \frac{3}{8} a_{j-1xxx} \right] \Lambda^{m-j} \]

Proof. By direct computing, Theorem 7 is derived. Now, substituting (18)-(20) into (30) and (31), then
\[ Y_1 = M Y \]
\[ Y_2 = W Y \]
where \( m = 0, 1, 2, \ldots \)

\[ M = \begin{pmatrix} 0 & -E & 0 & 0 \\ \frac{1}{2} (y_1, y_1) E & 0 & E & 0 \\ \frac{1}{2} (y_1, y_2) E & 0 & 0 & E \\ -A (y_1, y_1) - \frac{3}{8} (y_1, y_1)^2 E & \frac{1}{2} (y_1, y_2) E & -\frac{1}{2} (y_1, y_1) E & 0 \end{pmatrix} \]

and
\[ W = (\omega_y)_{m-4}, \quad m = 0, 1, 2, \ldots \]

\[ \omega_1 = \sum_{j=1}^{\infty} \left( \frac{1}{4} (\Lambda^{-j} y_1, z_1) \right) \Lambda^{m-j} \]
\[ \omega_2 = \sum_{j=1}^{\infty} \left( -\frac{1}{4} (\Lambda^{-j} y_1, z_1) \right) \Lambda^{m-j} \]
\[ \omega_3 = \sum_{j=1}^{\infty} \left( \frac{1}{4} (\Lambda^{-j} y_1, y_2) \right) \Lambda^{m-j} \]
\[ \omega_4 = \sum_{j=1}^{\infty} \left( \frac{1}{4} (\Lambda^{-j} y_1, y_1) \right) \Lambda^{m-j} \]

Theorem 8 On the Bargmann constrained equation (29), the evolution equation hierarchy (10) of the fourth-order eigenvalue problem (1) are nonlinearized as the following Hamilton canonical equation system
\[ Y_1 = \frac{\partial h}{\partial Z} \]
\[ Y_2 = \frac{\partial h}{\partial Y} \]

there \( m = 0, 1, 2, \ldots \), and
\[
\begin{align*}
\hat{h}_n &= -\frac{1}{2} \langle A^{m;1} y_1, y_1 \rangle + \langle A^n y_2, z_2 \rangle - \frac{1}{4} \langle A^n y_1, y_2 \rangle y_1 y_2 \\
&+ \frac{1}{2} \langle y_1, y_1 \rangle \langle A^n y_1, z_2 \rangle - \frac{1}{16} \langle A^n y_1, y_1 \rangle y_1^2 \\
&- \frac{1}{2} \langle A^{m;1} y_2, z_1 \rangle + \frac{1}{2} \langle A^n y_1, z_1 \rangle - \frac{1}{16} \langle A^n y_1, y_1 \rangle y_1^2 \\
&+ \langle A^n y_2, z_2 \rangle \langle A^{m;1} y_2, y_1 \rangle - \langle A^n y_2, y_1 \rangle \langle A^{m;1} y_2, z_1 \rangle \\
&- \langle A^n y_2, z_1 \rangle \langle A^{m;1} y_1, y_2 \rangle - \langle A^n y_2, y_2 \rangle \langle A^{m;1} y_1, z_1 \rangle \\
&+ \frac{1}{4} \sum_{j=1}^{N} \left( \langle A^{m;1} y_2, z_1 \rangle - \langle A^n y_2, y_1 \rangle \langle A^{m;1} y_2, z_1 \rangle \right) \\
&= \frac{1}{4} \sum_{j=1}^{N} \left( \langle A^{m;1} y_2, z_1 \rangle - \langle A^n y_2, y_1 \rangle \langle A^{m;1} y_2, z_1 \rangle \right)
\end{align*}
\]

**D. Integrable System of Hamilton Equations**

Set

\[
E^{(1)}_j = -\frac{1}{2} \left( y_j y_j - \frac{1}{8} \left( y_i y_i \right) y_j^2 ight) + \frac{1}{2} \left( y_j y_j - \frac{1}{2} \left( y_i y_i \right) y_j^2 \right) + \frac{1}{2} \left( y_j y_j - \frac{1}{4} \left( y_i y_i \right) y_j^2 \right) y_{ij} \\
E^{(2)}_j = \frac{1}{8} \Gamma^{(j,k)}_{ij} l, k = 1, 2
\]

there

\[
\Gamma^{(j,k)}_{ij} = \sum_{j=1}^{N} \frac{1}{\lambda_j - \lambda_l} \left( y_j y_j - \frac{1}{\lambda_j} y_j \right) \left( y_j y_j - \frac{1}{\lambda_l} y_j \right) \left( y_j y_j - \frac{1}{\lambda_m} y_j \right), l, k = 1, 2
\]

**Theorem 9.** (i) \( \{ E^{(m)}_j, j = 1, 2, \ldots, N; m = 0, 1, 2 \} \) are the involutive system, i.e. \( \{ E^{(m)}_j, E^{(n)}_j \} = 0 \) here, \( j, i = 1, 2, \ldots, N; m = 1, 2 \)

(ii) \( H_a = \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} (E^{(1)}_j + E^{(2)}_j) = \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} (E^{(1)}_j + E^{(2)}_j) \)

where

\[
h_a = \sum_{j=1}^{N} \lambda_j \left( E^{(1)}_j + E^{(2)}_j \right), m = 0, 1, 2, \ldots
\]

**Theorem 10.** (i) \( \{ h, E^{(1)}_j \} = 0 \)

where \( j = 1, 2, \ldots, N; k = 1, 2 \).

(ii) \( \{ h, h_a \} = 0 \).

**REFERENCES**


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