The Expected Value Model of Multiobjective Programming and its Solution Method Based on Bifuzzy Environment

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Abstract—In this paper, based on bifuzzy theory, we study the multiobjective programming problem under bifuzzy environment and present the expected value model to the problem. Furthermore, to the proposed model, the concepts of non-inferior solution are defined, and their relations are also discussed. According to practical decision-making process, a solution method, called the method of main objective function, has been studied, whose results can facilitate us to design algorithms to solve the bifuzzy multiobjective programming problem. Finally, a numerical example is given to explain the proposed method.

Index Terms—credibility theory, bifuzzy variable, multiobjective programming, expected value model

I. INTRODUCTION

The multiobjective programming problems are studied by many researchers such as [3], [15], [16],[18],[20]. For given multiobjective problem, its absolute optimal solutions which optimize each objective functions simultaneously usually do not exist, so we consider their non-inferior solutions, which are Pareto optimal solutions in common use. There are various types of uncertainties in the real-world problem. As we known, random phenomena is one class of uncertain phenomena which has been well studied. Based on the probability, stochastic multiobjective programming problems have been presented such as [1], [17]. Besides randomness, fuzziness is a basic type of subjective uncertainty initiated by [26]. Since the pioneering work of Zadeh, possibility theory was developed and extended by many researchers such as [2],[4],[7],[23],[21],[24]. Based on possibility theory, an axiomatic approach, called credibility theory [6], was studied extensively. From a measure-theoretic viewpoint, credibility theory provides a theoretical foundation for fuzzy programming [9] just like the role of probability theory in stochastic programming [5]. In a practical decision-making process, we often face a hybrid uncertain environment where linguistic and frequent nature coexist. For the examples of two fold uncertainty, we may refer to Liu [6], Liu [8], [10], Liu[11], Liu and Liu [13], Yazenin[22], Zhou[25]. To deal with this two fold uncertainty, it is required to employ bifuzzy theory[7]. The multiobjective programming under bifuzzy environment has not been developed well, therefore, following the idea of stochastic multiobjective programming, this paper devotes the bifuzzy multiobjective programming (BMOP) problems based on the random fuzzy theory. For the parameters of bifuzzy, we consider their expectation which convert the BMOP problem into the expected value model of bifuzzy multiobjective (EVBMOP) which is a deterministic multiobjective problem. By the deterministic problem above, we can obtain the expected value efficient solutions or expected value weakly efficient solutions to the BMOP problem. In actual problem, we usually need to distinguish between primary and secondary of the objective functions to the BMOP problem, so the method of main objective function is presented to solve the BMOP problem in this paper, which can covert the EVBMOP problems corresponding to the BMOP problem into the deterministic single objective programming problems whose optimal solutions are expected value weakly efficient solutions to the BMOP problems.

This paper is organized as follows. The next section provides a brief review on the related concepts and results in bifuzzy theory. Section 3 presents the BMOP problem and its expected value model. Furthermore, based on the expected value model, the expected value efficient solution and expected value weakly efficient solution to the BMOP are proposed, and their properties are discussed. To solve the BMOP problem, the method of main objective function is introduced in Section 4. Finally, Section 5 provides a summary of the main results of this paper.
II. Basic Concepts

Given a universe $\Gamma$, $\rho(\Gamma)$ is the power set of $\Gamma$, and a set function $\text{Pos}$ defined on $\rho(\Gamma)$ is called a possibility measure if it satisfies the following conditions[4]:

(1) $\text{Pos}(\emptyset) = 0$, $\text{Pos}(\Gamma) = 1$, and

(2) $\text{Pos}(\bigcup_{i\in I} A_i) = \sup_{i\in I} \text{Pos}(A_i)$ for any subclass $\{A_i | i \in I\}$ of $\rho(\Gamma)$.

The triplet $(\Gamma, \rho(\Gamma), \mathcal{C})$ is usually called a possibility space, which is called a pattern space by Nahimias [19].

In addition, a self-dual set function, called credibility measure, is defined as follows [12]:

$$\mathcal{C}(A) = \frac{1}{2} \left( 1 + \text{Pos}(A) - \text{Pos}(A^c) \right).$$

for any $A \in \rho(\Gamma)$, where $A^c$ is the complement of $A$.

A fuzzy variable $\xi$ is defined as a function from a credibility space $(\Gamma, \rho(\Gamma), \mathcal{C})$ to the set of real numbers. Based on credibility measure, the expected value of fuzzy variable $\xi$ is defined as follows [12]:

$$E[\xi] = \int_0^\infty \mathcal{C}(\xi \geq r) dr - \int_0^\infty \mathcal{C}(\xi \leq r) dr$$

(1)

provided that one of the two integrals is finite.

Given a credibility space $(\Gamma, \rho(\Gamma), \mathcal{C})$, which is complete, we obtain the definition of bifuzzy variable as follows:

**Definitions 2.1.**[7] Let $(\Gamma, \rho(\Gamma), \mathcal{C})$ be a credibility space. A map $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T: T \to \mathbb{R}^n$ is said to be an $n$-ary bifuzzy vector if for any Borel subset $B$ of $\mathbb{R}^n$, the function $\mathcal{C}(\gamma \in \Gamma, \xi(\gamma) \in B)$ is measurable with respect to $\gamma$. As $n = 1$, $\xi$ is called a bifuzzy variable.

**Definitions 2.2.**[7] Suppose $\xi$ is a bifuzzy variable, the expected value of $\xi$ is defined as the mathematical expectation of the fuzzy variable $E[\xi_1]$, i.e.,

$$E(\xi) = \int_\Gamma E[\xi_1] d\mathcal{C}(\gamma)$$

(2)

provided that the integrand $E[\xi_1]$ defined by Eq.(1) exists almost surely with respect to $\gamma$, and is integral.

From Eq. (2), we can provide the expectation of bifuzzy variable, i.e.,

$$E(\xi) = E_\gamma [E_\gamma [\xi_1(\gamma')]]$$

III BIFUZZY MULTIOBJECTIVE PROGRAMMING PROBLEMS

A. Expected Value Model of Bifuzzy Multiobjective Programming

Considering the bifuzzy multiobjective programming (BMOP) problem as follows:

$$\min_{x \in \mathbb{R}^n} \begin{cases} F(x, \xi) = (f_1(x, \xi), f_2(x, \xi), \ldots, f_m(x, \xi)) \\ \text{s.t.} \quad G(x, \xi) = (g_1(x, \xi), g_2(x, \xi), \ldots, g_n(x, \xi)) \leq 0 \\ H(x, \xi) = (h_1(x, \xi), h_2(x, \xi), \ldots, h_k(x, \xi)) = 0 \end{cases}$$

(BMOP)

where decision-making variable $x \in \mathbb{R}^n$, and $\xi$ is continuous bifuzzy variable.

For the BMOP problem, we assume the condition that $f_j(\xi'(\gamma'))$, $j = 1, 2, \ldots, m$, is borel measure function defined on measure space $(\Gamma \times \Gamma', \rho(\Gamma) \times \rho(\Gamma'), \mathcal{C} \times \mathcal{C}')$, hence, by the definition of bifuzzy variable, we can easily obtain that $f_j(x, \xi_1, \xi_2) = E_\gamma [f_j(x, \xi_1, \xi_2)(\gamma')]$ is a fuzzy variable for given $x \in \mathbb{R}^n$ and $\gamma \times \gamma' \in \Gamma \times \Gamma'$.

To solve the BMOP problem, based on the EVBMOP method, we present the expected value model of bifuzzy multiobjective programming (EVBMOP) problem which is a deterministic multiobjective programming problem as

$$\text{(EVBMOP)} \begin{cases} \min_{x \in \mathbb{R}^n} F(x, \xi) = (E(f_1(x, \xi)), E(f_2(x, \xi)), \ldots, E(f_m(x, \xi))) \\ \text{s.t.} \quad E(G(x, \xi)) = (E(g_1(x, \xi)), E(g_2(x, \xi)), \ldots, E(g_n(x, \xi))) \leq 0 \\ E(H(x, \xi)) = (E(h_1(x, \xi)), E(h_2(x, \xi)), \ldots, E(h_k(x, \xi))) = 0 \end{cases}$$

where

$$D' = \left\{ x \in \mathbb{R}^n | E(G(x, \xi)) = (E(g_1(x, \xi)), E(g_2(x, \xi)), \ldots, E(g_n(x, \xi))) \leq 0, \quad E(H(x, \xi)) = (E(h_1(x, \xi)), E(h_2(x, \xi)), \ldots, E(h_k(x, \xi))) = 0 \right\}$$

**Theorem 3.1.** Let $\xi$ be a bifuzzy variable, $F(x, t)$ and $G(x, t)$ be convex vector function on $x$ for any given $t$. Furthermore, $H(x, t)$ is also convex vector function. In addition, for any given $x_1$ and $x_2$, $F(x_1, t)$ and $F(x_2, t)$ (correspondingly, $G(x_1, t)$ and $G(x_2, t)$ are comonotonic on $t$, then the EVBMOP problem is a convex programming problem.

**Proof.** To prove the theorem, it is sufficient to illuminate that $E[F(x, \xi)]$ is a convex vector function and the feasible region $D$ is a convex set. By the assumed conditions, for any given $t$, we can obtain:

$$F(x_1, t) + (1 - \lambda)F(x_2, t),$$

for any $\lambda \in [0, 1]$, and $x_1, x_2 \in \mathbb{R}^n$.

It is evident that the following inequality holds for bifuzzy variable $\gamma \times \gamma' \in \Gamma \times \Gamma'$.

$$\lambda E[F(x_1, \xi_1, \xi_2(\gamma'))] + (1 - \lambda) E[F(x_2, \xi_1, \xi_2(\gamma'))] \leq \lambda E[F(x_1, \xi_1, \xi_2(\gamma))] + (1 - \lambda) E[F(x_2, \xi_1, \xi_2(\gamma))].$$

From the comonotonic properties, taking the expectation of fuzzy variable to the inequality above, we have

$$E_\gamma [F(\lambda x_1 + (1 - \lambda) x_2, \xi_1, \xi_2)] \leq \lambda E[F(x_1, \xi_1, \xi_2)] + (1 - \lambda) E[F(x_2, \xi_1, \xi_2)]$$

which shows $E[F(x, \xi)]$ is a convex vector function.

Next we will illuminate that the feasible region $D$ is a convex set. If $x_1, x_2 \in D$, it follows from the convexity of vector function $G(x, t)$ that

$$G(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda G(x_1, t) + (1 - \lambda) G(x_2, t)$$

for any $t$ and $\lambda \in [0, 1]$. 

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Similarly, by the linear properties of fuzzy variable, we can obtain:
\[ G(\lambda x_1 + (1 - \lambda)x_2, \xi) \]
\[ \leq \lambda G(x_1, \xi) + (1 - \lambda)G(x_2, \xi) \]
Using the same method above, we can obtain:
\[ E[G(\lambda x_1 + (1 - \lambda)x_2, \xi)] \]
\[ \leq \lambda E[G(x_1, \xi)] + (1 - \lambda)E[G(x_2, \xi)] \quad (6) \]

On the other hand, because \( H(x, t) \) is linear vector function, we can obtain
\[ H(\lambda x_1 + (1 - \lambda)x_2, \xi) = \lambda H(x_1, \xi) + (1 - \lambda)H(x_2, \xi). \]

Similarly, it follows from the comonotonic properties that
\[ E[H(\lambda x_1 + (1 - \lambda)x_2, \xi)] = \lambda E[H(x_1, \xi)] + (1 - \lambda)E[H(x_2, \xi)] = 0 \quad (7) \]

Obviously, by Eq.(6) and Eq.(7), it shows that feasible region \( D \) is a convex set.

Hence, the EVBMOP problem is a convex programming. The proof is complete.

B. Three solution Concepts and Their Relations

Definitions 3.1. For the EVBMOP problem, if \( x^* \in D \) we say that \( x^* \) is the expected value absolute optimal solution to the BMOP problem whose solution set is denoted \( D_{ab} \) if it satisfies the following conditions:
\[ E[F(x^*, \xi)] \leq E[F(x, \xi)] \]

namely,
\[ E[f_j(x^*, \xi)] \leq E[f_j(x, \xi)] \quad \text{for all } j = 1, 2, \ldots, p. \]

Definitions 3.2. For the EVBMOP problem, if \( x^* \in D \) we say that \( x^* \) is the expected value efficient solutions to the BMOP problem whose solution set is denoted \( D_{pa} \) if it satisfies the following conditions: there doesn’t exist \( x \in D \) such that
\[ E[F(x, \xi)] \leq E[F(x^*, \xi)] \]

namely,
\[ E[f_j(x, \xi)] \leq E[f_j(x^*, \xi)] \quad \text{for all } j = 1, 2, \ldots, p. \]

and there must exist some \( j_0 \) at least such that
\[ E[f_{j_0}(x, \xi)] < E[f_{j_0}(x^*, \xi)]. \]

Definitions 3.3. For the EVBMOP problem, if \( x^* \in D \) we say that \( x^* \) is the expected value weakly efficient solutions to the BMOP problem whose solution set is denoted \( D_{wpa} \), and if it satisfies the following conditions: there doesn’t exist \( x \in D \) such that
\[ E[F(x, \xi)] < E[F(x^*, \xi)] \]

Theorem 3.2. \( D_{ab} \subseteq D_{pa} \subseteq D_{wpa} \subseteq D \)

Proof. We first prove that \( D_{ab} \subseteq D_{pa} \). If \( D_{ab} = \phi \), then the result is immediate. If not, suppose that \( x^* \in D_{ab} \), and \( x^* \notin D_{pa} \), then, by the definition of the expected value efficient solution, their must exist \( \bar{x} \in D \), such that
\[ E[F(\bar{x}, \xi)] \leq E[F(x^*, \xi)] \]


\[
E[f_j(x^*, \xi)] \leq E[f_j(x, \xi)] \quad \text{for all } j = 1, 2, \ldots, p, \]

and their exists \( j_0 \) at least such that
\[ E[f_{j_0}(x^*, \xi)] < E[f_{j_0}(x, \xi)], \]

which implies the contradiction with \( x^* \in D_{ab} \). Hence,
\[ D_{ab} \subseteq D_{pa}. \]

Then we prove that \( D_{pa} \subseteq D_{wpa} \). If \( x^* \in D_{pa} \), and \( x^* \notin D_{wpa} \), then, by the definition of expected value weakly efficient solution, their must exist \( \bar{x} \in D \), such that
\[ E[F(\bar{x}, \xi)] < E[F(x^*, \xi)], \]

namely,
\[ E[f_j(\bar{x}, \xi)] < E[f_j(x^*, \xi)] \quad \text{for all } j = 1, 2, \ldots, p. \]

Thus, we can obtain:
\[ E[F(\bar{x}, \xi)] \leq E[F(x^*, \xi)] \]

which implies \( \bar{x} \notin D_{pa} \). By the previous assumption, we can obtain the contradiction with \( x^* \in D_{pa} \). Hence,
\[ D_{pa} \subseteq D_{wpa}. \]

It follows from the definition of the expected value weakly efficient solution that \( D_{wpa} \subseteq D \), which proves the desired theorem.

Theorem 3.3. (1) If \( D_{ab} \neq \phi \), then \( D_{ab} = D_{pa} \).

(2) If \( H(x, t) \) is linear vector function, \( F(x, t) \) and \( G(x, t) \) are strict convex vector function on \( x \).

Furthermore, for any given \( x_1 \) and \( x_2 \), \( F(x_1, t) \) and \( F(x_2, t) \) (correspondingly, \( G(x_1, t) \) and \( G(x_2, t) \) are comonotonic on \( t \), then we can obtain \( D_{ab} \supseteq D_{pa} \).

Proof. (1) It follows from Theorem 3.2 that we need only to prove \( D_{ab} \supseteq D_{pa} \). If \( x^* \in D_{pa} \), and \( x^* \notin D_{ab} \), since \( D_{ab} = \phi \), their must exist \( \bar{x} \in D \), by the definition of expected value absolute optimal solution, we can obtain
\[ E[F(x, \xi)] < E[F(x^*, \xi)]. \]

Since \( x^* \notin D_{ab} \), we have
\[ E[F(x, \xi)] \neq E[F(x^*, \xi)]. \]

It follows from the inequality above that
\[ E[F(x, \xi)] < E[F(x^*, \xi)], \]

which is a contradiction with \( x^* \in D_{pa} \). Hence, 
\[ D_{ab} \supseteq D_{pa} \], which implies the required conclusion.

(2) It follows from Theorem 3.2 that we need only to prove \( D_{pa} \supseteq D_{wpa} \). If \( x^* \in D_{wpa} \), and \( x^* \notin D_{pa} \), we know that their must exist \( \bar{x} \in D \), and \( \bar{x} \neq x^* \), such that
\[ E[F(x, \xi)] < E[F(x^*, \xi)]. \]

By the assumed conditions and Theorem 3.1, we can obtain that \( D \) is a convex set, hence, 
\[ \alpha x + (1 - \alpha)x^* \in D, \]

for any given \( \alpha \in (0, 1) \). Since
\( F(x, \xi) \) is strict convex vector function on \( D \), and is also comonotonic, by noting the inequality just given, it easy to know that

\[
E[F(ax+(1-a)x^*)] < aE(F(x, \xi)] + (1-a)E[F(x^*, \xi)] < E[F(x^*, \xi)],
\]

which is a contradiction with \( x^* \in D_{\text{ups}} \). Thus, \( D_{\text{pa}} \supset D_{\text{ups}} \), which proves the required theorem.

IV SOLUTION METHOD

A. Expected Value Model of Bifuzzy Multiobjective Programming

In real-world problems, we just need to consider the main objective function to some real-life problems, therefore, a type of method, called the method of main objective function, is presented in the following. Without any loss of generality, let \( f_i(x, \xi) \) be regarded as main objective function to the BMOP problem, and wish the expectation of the other objective functions \( f_j(x, \xi) \), \( j=2,3,\ldots,p \), satisfy the following constraint-conditions:

\[ E[f_j(x, \xi)] \leq \alpha_j, \quad j=2,3,\ldots,p. \]

Then the BMOP problem can be transformed into the following SOP problem

\[
\min_{x \in D} E[f_1(x, \xi)] \quad (8)
\]

where

\[ \tilde{D} = \{ x \in D | E[f_i(x, \xi)] \leq \alpha_i, \quad j=2,3,\ldots,p \}, \]

whose optimal solution set is denoted as \( D_{\text{sub}} \).

Obviously, the constraint set \( \tilde{D} \) to problem (8) is a new set which is added into several constraint conditions:

\[ E[f_j(x, \xi)] \leq \alpha_j, \quad j=2,3,\ldots,p. \]

Thus, we employ the method of solving the nonlinear programming which is a linear problem in particular to solve the transformed SOP problem whose optimal solution is the non-inferior solution to the BMOP problem verified by the following theorem.

**Theorem 4.1.** \( D_{\text{sub}} \subset D_{\text{ups}} \)

**Proof.** If \( x^* \in D_{\text{sub}} \) and \( x^* \in D_{\text{ups}} \), then, by the definition of expected value weakly efficient solution, there must exist some \( \bar{x} \in D \), such that

\[ E[f_j(\bar{x}, \xi)] < E[f_j(x^*, \xi)] \quad \text{for all } j=1,2,\ldots,p. \]

Since \( x^* \in \tilde{D} \), we have

\[ E[f_j(x^*, \xi)] \leq \alpha_j, \quad j=2,3,\ldots,p, \]

It follows from inequality above that

\[ E[f_j(\bar{x}, \xi)] \leq E[f_j(x^*, \xi)] \leq \alpha_j, \quad j=2,3,\ldots,p, \]

which illuminates \( \bar{x} \in \tilde{D} \), i.e., \( \bar{x} \) is the feasible solution to SOP problem, therefore, it is easy to know

\[ E[f_j(\bar{x}, \xi)] < E[f_j(x^*, \xi)]. \]

which is a contradiction with the previous hypothesis that \( x^* \in D_{\text{sub}} \). Hence, \( x^* \in D_{\text{ups}} \), which implies the required conclusion.

**Theorem 4.2.** Without any loss of generality, assuming that \( f_i(x, \xi) \) is the main objective function, if \( H(x, \xi) \) is linear vector function, \( F(x, \xi) \) and \( G(x, \xi) \) are strict convex vector function on \( x \). Furthermore, for any given \( x_1 \) and \( x_2 \), \( F(x_1,t) \) and \( F(x_2,t) \) (correspondingly, \( G(x_1,t) \) and \( G(x_2,t) \) are comonotonic on \( t \), then \( D_{\text{sub}} \subset D_{\text{pa}} \). In addition, if \( D_{\text{sub}} \neq \emptyset \), we can obtain:

\[ D_{\text{sub}} \subset D_{\text{ab}}. \]

**Proof.** It follows from the assumed conditions that

\[ E[f_1(x, \xi)] \]

is strict convex function, so the optimal solution to SOP problem must be unique. If \( x^* \) is the unique optimal solution to the SOP problem, and \( x^* \not\in D_{\text{pa}} \), there must exist \( x \in D \) and \( x \neq x^* \) such that

\[ E[F(x, \xi)] \leq E[F(x^*, \xi)], \]

i.e.,

\[ E[f_j(\bar{x}, \xi)] \leq E[f_j(x^*, \xi)], \quad j=1,2,3,\ldots,p. \]

Obviously, there must exist some \( j_0 \) such that

\[ E[f_{j_0}(\bar{x}, \xi)] \leq E[f_{j_0}(x^*, \xi)]. \]

It is easy to know that \( j_0 \neq 1 \). In fact, if \( j_0 = 1 \), then \( E[f_{j_0}(\bar{x}, \xi)] \leq E[f_{j_0}(x^*, \xi)] \), which is a contradiction with the previous hypothesis that \( x^* \) is the optimal solution. Hence, we can obtain that

\[ E[f_{j_0}(\bar{x}, \xi)] = E[f_{j_0}(x^*, \xi)]. \]

Since \( x^* \in \tilde{D} \), we have

\[ E[f_j(x^*, \xi)] \leq \alpha_j, \quad j=2,3,\ldots,p. \]

Hence,

\[ E[f_j(\bar{x}, \xi)] \leq E[f_j(x^*, \xi)] \leq \alpha_j, \quad j=2,3,\ldots,p, \]

which shows that \( \bar{x} \in \tilde{D} \). It follows by noting that \( E[f_{j_0}(\bar{x}, \xi)] = E[f_{j_0}(x^*, \xi)] \) that \( \bar{x} \) is the optimal solution to the SOP problem, which is a contradiction with the previous hypothesis that the optimal solution to SOP problem is unique. Thus, \( x^* \in D_{\text{pa}} \), which verifies the required conclusion.

Furthermore, if \( D_{\text{ab}} \neq \emptyset \), it follows by the Theorem 3.3(1) that \( D_{\text{ab}} \subset D_{\text{pa}} \). By the conclusion that \( D_{\text{ab}} \subset D_{\text{pa}} \) just discussed above, we can obtain \( D_{\text{sub}} \subset D_{\text{ab}} \) easily. The proof of the theorem is complete.

**Remark 1.** Generally, we take the limit value \( \alpha_i \geq j_0 = \min_{x \in \tilde{D}} E[f_j(x, \xi)] \), by which can make some optimal solutions of \( E[f_j(x, \xi)] \) in \( \tilde{D} \). Of course, in real-world problem, it is difficult to solve \( \min_{x \in \tilde{D}} E[f_j(x, \xi)], \quad j=2,3,\ldots,p \). So we usually take
the \( \alpha_i \) according to the actual demand. Furthermore, if the \( \alpha_i \) is not well-found, then the feasible sets \( \tilde{D} \) may be empty set, which can’t get the optimal solution of SOP problem, that is, we can’t obtain the expected value non-inferior solutions to the BMOP problem, so we can take the following measure which can avoid that \( \tilde{D} \) is empty set:

\[
\alpha_j = \mathbb{E}[f_j(x^0, \xi)], \quad j = 2, 3, \ldots, p,
\]

for any given \( x^0 \in \tilde{D} \), which can guarantee that one solution at least, i.e., there exist \( x^0 \in \tilde{D} \) at least. Furthermore, the optimal solutions of the SOP problem by the measure proposed above must be the expected value weakly efficient solution of the BMOP problem, and it may be not satisfying, but it is the practical technique to deal with real-life problem frequently.

### B. Expected Value Model of Bifuzzy Multiobjective Programming

In particular, if the bifuzzy variable \( \xi \) involved in the problem (8) is a discrete one, we will illuminate how to calculate the \( \mathbb{E}[f_1(x, \xi)] \). Assume that the bifuzzy variable \( \xi \) is a discrete one such that \( \gamma \) is a discrete fuzzy variable taking on finite number of values with possibility \( \mu_i, \quad i = 1, 3, \ldots, N \), respectively, and satisfying \( \max_i \mu_i = 1 \), \( j = 1, 3, \ldots, N \), and for each \( i \), fuzzy variable \( \xi_{\gamma_i} \) taking on the following values

\[
\xi_{\gamma_i}(\gamma_{i1}) \quad \text{with possibility } \mu_{i1} > 0;
\]

\[
\xi_{\gamma_i}(\gamma_{i2}) \quad \text{with possibility } \mu_{i2} > 0;
\]

\[ \cdots \]

\[
\xi_{\gamma_i}(\gamma_{iN_i}) \quad \text{with possibility } \mu_{iN_i} > 0;
\]

\[
\ldots
\]

\[
\xi_{\gamma_i}(\gamma_{2N_i}) \quad \text{with possibility } \mu_{2N_i} > 0;
\]

\[ \cdots \]

\[
\xi_{\gamma_i}(\gamma_{i1}) \quad \text{with possibility } \mu_{i1} > 0;
\]

\[
\xi_{\gamma_i}(\gamma_{i2}) \quad \text{with possibility } \mu_{i2} > 0;
\]

\[ \cdots \]

\[
\xi_{\gamma_i}(\gamma_{iN_i}) \quad \text{with possibility } \mu_{iN_i} > 0;
\]

\[ \cdots \]

\[
\xi_{\gamma_i}(\gamma_{N_i}) \quad \text{with possibility } \mu_{N_i} > 0;
\]

\[ \cdots \]

\[
\xi_{\gamma_i}(\gamma_{N_i}) \quad \text{with possibility } \mu_{N_i} > 0;
\]

It is easy to obtain the expectation of fuzzy variable \( f_1(x, \xi_{\gamma}) \) as follows:

\[
f_1(x, \xi_{\gamma}) = \mathbb{E}_\gamma[f_1(x, \xi_{\gamma}(\gamma'))] = \sum_{i=1}^{N_i} p_{ij} f_1(x, \xi_{\gamma_i}(\gamma_{ij})) \quad (9)
\]

where \( p_{ij} \) are the weights of fuzzy variable \( f_1(x, \xi_{\gamma_i}(\gamma_{ij})) \) calculated by the following formulas [14]:

\[
p_{ij} = \frac{1}{2} (\max_{k \in \alpha_i} \mu_k - \max_{k \in \beta_i} \mu_k) + \frac{1}{4} \max_{k \in \alpha_i} \mu_k - \max_{k \in \beta_i} \mu_k \quad (10)
\]

where \( \mu_\alpha = \mu_{N_i+1} = 0, \quad i = 1, 2, \ldots, N \), \( j = 1, 2, \ldots, N_i \), and satisfies the following constrains:

\[
p_{ij} \geq 0, \quad \sum_{i=1}^{N_i} p_{ij} = \max_{i=1}^{N_i} \mu_{ij} = 1.
\]

By the Eq.(2), the expectation of bifuzzy variable \( f_1(x, \xi_{\gamma}) \) are given in the following

\[
\mathbb{E}[f_1(x, \xi)] = \mathbb{E}_\gamma[f_1(x, \xi_{\gamma})] = \sum_{i=1}^{N} p_i f_1(x, \xi_{\gamma_i}) \quad (11)
\]

where \( p_i \) are the weights of fuzzy variable \( f_1(x, \xi_{\gamma_i}) \) calculated similarly by the Eq.(10).

**Example 4.1.** Solving the following bifuzzy multiobjective programming

\[
\min \{f_1(x, \xi), f_2(x, \xi)\}
\]

\[
= \{5x_1 + 7x_2 - 4x_3 + 2\xi_1, -2x_1 + 3x_2 + 8x_3 - 3\xi_2\}
\]

\[
s.t. \quad x_1 + 2x_2 - 3x_3 \leq 5 \quad (12)
\]

where \( f_1(x, \xi) \) is the main objective function, and the limit value of \( \mathbb{E}[f_2(x, \xi)] \) is 4.4, i.e.,

\[
\mathbb{E}[f_2(x, \xi)] \leq \alpha_2 = 4.4 .
\]

In addition, \( \xi \) is the discrete bifuzzy variable defined as

\[
\xi = \begin{cases} X_1, & \text{with possibility } 3/5 \\ X_2, & \text{with possibility } 1/4 \\ X_3, & \text{with possibility } 1 .
\end{cases}
\]

Here the fuzzy variable \( X_1 \) assumes the value 3, 4, 5 with the possibility 1/4, 3/4, and 1, respectively; \( X_2 \) assumes the value 1, 2, 3 with the possibility 5/12, 1 and 7/12, respectively; and \( X_3 \) assumes the value 6, 8, 10 with the possibility 2/7, 1/7 and 1, respectively.

According to the method of main objective function discussed above, the problem (12) can be transformed into the following single objective problem

\[
\min \mathbb{E}[f_1(x, \xi)] = \mathbb{E}[5x_1 + 7x_2 - 4x_3 + 2\xi_1]
\]

\[
s.t. \quad \mathbb{E}[-2x_1 + 3x_2 + 8x_3 - 3\xi_2] \leq 4.4
\]

\[
x_1 + 2x_2 - 3x_3 \leq 5
\]

\[
-7x_1 - 3x_2 + 7x_3 \leq 3
\]

\[
11x_1 + 5x_2 - 6x_3 \leq 10.
\]

We can obtain the following results.
\[
\begin{align*}
    f_1(x, \xi^n_1, \nu_1' & ) = 5x_1 + 7x_2 - 4x_3 + 6, \\
    & f_1(x, \xi^n_2, \nu_2' & ) = 5x_1 + 7x_2 - 4x_3 + 8, \\
    & f_1(x, \xi^n_3, \nu_3' & ) = 5x_1 + 7x_2 - 4x_3 + 10.
\end{align*}
\]

It is easy to know
\[
f_1(x, \xi^n_1, \nu_1') \leq f_1(x, \xi^n_2, \nu_2') \leq f_1(x, \xi^n_3, \nu_3').
\]

Therefore, we can obtain the distribution function of fuzzy variable \( f_1(x, \xi^n_1, \nu_1') \) as
\[
\begin{align*}
    \{5x_1 + 7x_2 - 4x_3 + 6, & \quad \text{with possibility } 1/4, \\
    5x_1 + 7x_2 - 4x_3 + 8, & \quad \text{with possibility } 3/4, \\
    5x_1 + 7x_2 - 4x_3 + 10, & \quad \text{with possibility } 1
\end{align*}
\]

with weights \( p_{11} = 1/8, p_{12} = 1/4, \text{and } p_{13} = 5/8 \), respectively, which are calculated by Eq.(10).

It follows the Eq.(9) that
\[
f_1(x, \xi^n_1, \nu_1') = \sum_{j=1}^{3} p_j f_1(x, \xi^n_1, \nu_1')
\]
\[
= \frac{1}{8} (5x_1 + 7x_2 - 4x_3 + 6) + \frac{3}{4} (5x_1 + 7x_2 - 4x_3 + 8) + \frac{5}{8} (5x_1 + 7x_2 - 4x_3 + 10)
\]
\[
= 5x_1 + 7x_2 - 4x_3 + 9,
\]

whose possibility is 3/5.

Similarly, we can obtain
\[
\begin{align*}
    f_1(x, \xi^n_2, \nu_2') & = 5x_1 + 7x_2 - 4x_3 + 25/6, \\
    f_1(x, \xi^n_3, \nu_3') & = 5x_1 + 7x_2 - 4x_3 + 132/7,
\end{align*}
\]

with the possibility 1/4 and 1, respectively.

Obviously,
\[
f_1(x, \xi^n_1, \nu_1') \leq f_1(x, \xi^n_2, \nu_2') \leq f_1(x, \xi^n_3, \nu_3').
\]

Hence, without any loss of generality, the distribution function of fuzzy variable \( f_1(x, \xi^n_1, \nu_1') \) is the following
\[
f_1(x, \xi^n_1, \nu_1') = \{5x_1 + 7x_2 - 4x_3 + 25/6, \quad \text{with possibility } 1/4, \\
5x_1 + 7x_2 - 4x_3 + 9, \quad \text{with possibility } 3/5, \\
5x_1 + 7x_2 - 4x_3 + 132/7, \quad \text{with possibility } 1
\]

with weights \( p_{11} = 1/8, p_{12} = 7/40, \text{and } p_{13} = 7/10 \), respectively, which are calculated by Eq.(10).

By the Eq.(11), we can deduce
\[
E[f_1(x, \xi)] = E[f_1(x, \xi^n_1, \nu_1')] = \sum_{j=1}^{3} p_j E_X f_1(x, \xi^n_1)
\]
\[
= \frac{1}{8} (5x_1 + 7x_2 - 4x_3 + 25/6) + \frac{3}{4} (5x_1 + 7x_2 - 4x_3 + 9) + \frac{5}{8} (5x_1 + 7x_2 - 4x_3 + 132/7)
\]
\[
= 5x_1 + 7x_2 - 4x_3 + 20.953.
\]

Using the same method, we can obtain the expectation of the bifuzzy variable \( f_2(x, \xi) \) as follows
\[
E[f_2(x, \xi)] = E[f_2(x, \xi^n_1, \nu_1')] = \sum_{j=1}^{3} p_j E_X f_2(x, \xi^n_1)
\]
\[
= -2x_1 + 3x_2 + 8x_3 - 23.25.
\]

Therefore, problem (13) is equivalent to the following problem:
\[
\begin{align*}
    \min & \{5x_1 + 7x_2 - 4x_3 + 20.953, \\
    & x + 3x_2 - 4x_3, \\
    & x + 2x_2 - 3x_3 \leq 5 \\
    & -7x_1 - 3x_2 + 7x_3 \leq 3 \\
    & 11x_1 + 5x_2 - 6x_3 \leq 10,
\end{align*}
\]

whose optimal solution is \((x_1, x_2, x_3) = (-0.4286, 0, 0)\) solved by LINGO software. Furthermore, we can obtain that \(x^* = (-0.4286, 0, 0)\) is the expected value weakly efficient solution to problem (12) by the Theorem 4.1.

V CONCLUSIONS

In this study, we mainly concerned the expected value model and the solution method of the multiobjective programming problem under bifuzzy environment. We first presented a new type of bifuzzy multiobjective programming problem. As we known, the non-inferior solutions play important role to multiobjective problem, so the expected value non-inferior solutions to the BMOP problem are presented and their relations are also studied. In addition, a solution method, called the method of main objective function, was discussed, which facilitated us to design algorithms to solve the BMOP problem.

REFERENCES


